On the propagation of nonlinear waves over a viscoelastic foundation

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It has long been evidenced that muddy seafloors in coastal areas can cause strong dissipation of ocean waves [1]. As waves propagate over a deformable seafloor, the wave-induced pressure force causes the displacement of the mud, which in turn modifies the flow field, resulting in the attenuation of the wave. The associated wave energy loss can be considerable. This wave transformation effect has been observed at various locations around the world. For example, in the Gulf of Mexico and in southwest coast of India, where muddy areas reduce severity of large waves and are used by fishermen as emergency harbours during severe storms [2].

1. INTRODUCTION

A growing number of offshore structures (floating and bottom-mounted wind turbines, floating solar panels, etc.) are being installed in coastal areas as part of renewable energy projects. The presence of mud, due to its ability to dampen ocean waves, can significantly reduce the hydrodynamic loads on these structures and broaden their operational limits. The new technology of elastic wave carpet [3], mimicking the damping features of mud, has the potential to be employed in combination with wind and wave energy converters, and thereby recast the existing implementation of the renewable energy systems. In addition, the underwater carpet moves the attached pumps to convert the produced hydraulic pressure into power. With a global increase in offshore structures, the problem of wave interaction with deformable seafloors is becoming an increasingly important economic and environmental societal challenge.

Many theories have been proposed in the past to describe the mechanism of surface wave dissipation by a deformable seafloor owing to different assumptions about the rheology of the mud sediment and the character of its response [4]. These include combinations of elastic, viscous, porous, plastic and Bingham plastic materials. Majority of the previous studies are confined to the waves of small amplitudes, whereas seabed effects are more significant as surface waves propagate into shallow waters where the wave nonlinearity increases and is expected to be important. The purpose of this abstract is to introduce the theoretical model for the interaction of nonlinear water waves with a deformable seafloor and describe briefly some characteristics of the associated wave transformations.

2. THE GREEN-NAGHDI EQUATIONS

The equations governing the motion of the fluid in this study are provided by the nonlinear Green-Naghdi theory (GN hereafter). The GN theory was originally developed based on the theory of directed fluid sheets by Green & Naghdi [5] and is used to describe wave propagation in deep or shallow waters over an arbitrary seafloor. The GN equations are classified based on the level of the functions used to prescribe the distribution of the vertical velocity along the water column. The Level I GN equations, also known as the restricted theory, assume a linear distribution of the vertical velocity along the water column. This assumption, which is the only one made about the kinematics of the fluid sheet, results in the horizontal velocities being invariant in the vertical direction for an incompressible fluid. The GN equations has been successfully applied to the problems of wave structure interaction [6].

The flow of incompressible inviscid fluid over a deformable seafloor is considered in a two-dimensional Cartesian reference frame in which the x axis is pointing to the right, the y axis is directed upward, and its origin is located on the undisturbed free surface of the fluid (see figure 1). The governing equations are presented in dimensionless form after using the density of the fluid ρ, the fluid depth h and acceleration due to gravity g as a dimensionally independent set:

$$\eta_{,t} + [(1+\eta-\alpha)u]_{,x} = \alpha_{,t}, \tag{1}$$

$$u_{,t} + uu_{,x} + \eta_{,x} = -\frac{1}{6}\Big\{(2\eta+\alpha)_{,x}\ddot{\alpha} + (4\eta-\alpha)_{,x}\ddot{\eta} + (1+\eta-\alpha)(\ddot{\alpha}+2\ddot{\eta})_{,x}\Big\}, \tag{2}$$

$$\bar{p} = \frac{1}{2}(1+\eta-\alpha)(\ddot{\alpha}+\ddot{\eta}+2). \tag{3}$$

Here $u(x,t)$ is the depth-averaged horizontal fluid velocity, $\eta(x,t)$ is the free surface elevation measured from the still water level, $\alpha(x,t)$ is the displacement of the bottom measured from the state of rest and $\bar{p}$ is the fluid pressure on the seafloor. In our notation, subscript after comma denotes the partial differentiation with respect to the given variable and superimposed dot specifies the total time derivative. By virtue of the GN theory, the exact boundary conditions on the free surface and the seafloor are automatically satisfied by equations (1)-(3).

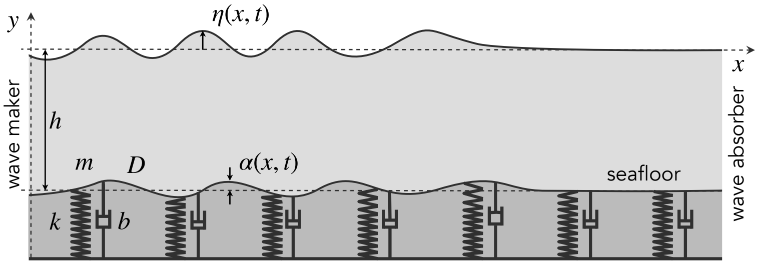

Figure 1: Scheme of wave interaction with deformable elastic sheet lying on a viscoelastic foundation.

The deformable bottom is represented by an inifite elatic sheet resting on the viscoelastic foundation. The dynamic pressure $\tilde{p}$ induced by elastic deformation of the sheet is given by the Euler-Bernoulli theory as

$$\tilde{p} = m\alpha_{,tt} + D\alpha_{,xxxx} + b\alpha_{,t} + k\alpha + 1, \tag{4}$$

where m and D are the unit mass and flexural rigidity of the plate, b and k are the viscous damping and stiffness of the elastic foundation per unit area, respectivey. The unity on the right-hand side of equation (4) accounts for the hydrostatic pressure from the water column, creating the initial uniform tension in the viscoelastic foundation. The balance of hydrodynamic and elastic pressures at the bottom ($\bar{p} = \tilde{p}$) gives the complementary governing equation:

$$m\alpha_{,tt} + D\alpha_{,xxxx} + b\alpha_{,t} + k\alpha - \frac{1}{2}(1+\eta-\alpha)(\ddot{\alpha}+\ddot{\eta}) - \eta + \alpha = 0. \tag{5}$$

The main difficulty in solving the system of equations (1), (2) and (5) numerically is the presence of the time derivative of bottom displacement $\alpha(x,t)$ in the mass equation (1). In the case of the rigid bottom, the time derivative of $\eta(x,t)$ could be eliminated by applying mass continuity equation (1) to momentum equation (2). As a result, the momentum equation is formulated in terms of time derivatives of horizontal velocity with the coefficients and the right-hand side depending on spatial derivatives of the unknown functions, see [6]. In the case of deformable bottom, however, we modify this approach by elaborating the mass continuity equation (1) to:

$$\ddot{\eta} - \ddot{\alpha} = -\phi u_{,xt} + \phi(u_{,x}^2 - uu_{,xx}). \tag{6}$$

Here ϕ denotes the thickness of the fluid layer $1+\eta-\alpha$. By use of equation (6), we transform the momomentum equation (2) and pressure balance equation (5) to eliminate the second-order full time derivatives of η and α. As a result, we obtain the equations containing the partial time derivativies of the unknown horizontal velocity $u(x,t)$ and plate deflection $\alpha(x,t)$:

$$(1+\alpha_{,x}\eta_{,x}+\tfrac{\phi\alpha_{,xx}}{2})u_{,t} - \phi\phi_{,x}u_{,xt} - \tfrac{\phi^2}{3}u_{,xxt} + (\phi u_{,x}+2\eta_{,x}u)\alpha_{,xt} + \phi u\alpha_{,xxt} + \eta_{,x}\alpha_{,tt} + \tfrac{\phi}{2}\alpha_{,xtt} = -\overline{Y}, \tag{7}$$

$$\phi\alpha_{,x}u_{,t} - \tfrac{1}{2}\phi^2 u_{,xt} + b\alpha_{,t} + 2\phi u\alpha_{,xt} + (\phi+m)\alpha_{,tt} = -\underline{Y}, \tag{8}$$

where functions $\overline{Y}$ and $\underline{Y}$, including only the spatial derivatives of the unknown functions, account for the effects of the free surface and deformable bottom, respectively:

$$\overline{Y} = \eta_{,x} + (\eta_{,x}\alpha_{,xx} + \frac{1}{2}\phi\alpha_{,xxx})u^2 + (1 + \alpha_{,x}\eta_{,x} + \frac{3}{2}\phi\alpha_{,xx})uu_{,x} + \frac{1}{3}\phi^2(u_{,x}u_{,xx} - uu_{,xxx}) + \\ + \phi\eta_{,x}(u_{,x}^2 - uu_{,xx}) + \frac{1}{2}\phi\alpha_{,x}(u_{,x}^2 + uu_{,xx}) \tag{9}$$

$$\underline{Y} = \eta - \alpha + k\alpha + D\alpha_{,xxxx} + \frac{1}{2}\phi^2(u_{,x}^2 - uu_{,xx}) + \phi u(\alpha_{,x}u_{,x} - u\alpha_{,xx}). \tag{10}$$

In the case of $\alpha \equiv 0$, equation (7) describes wave propagation over a flat rigid bottom. Equations (1), (7) and (8) form the full set of partial differential equations in the entire flow domain, which can be discretized and solved simultaneously for the unknowns. Note, that the last two equations (7) and (8) contain only the time derivative of the horizontal velocity $u(x,t)$ and bottom displacement $\alpha(x,t)$ and hence can be processed independently from the mass equation (1).

3. NUMERICAL SOLUTION

To solve equations (1), (7) and (8) numerically, we use an explicit time stepping scheme, consisting of the first-order Euler method applied twice. The spatial discretization of the derivatives is performed using the second-order accurate central difference formulas for the nodes inside the domain. As a result of discretization on a finite uniform grid of $N+2$ nodes, the set of equations (1), (7) and (8) is reduced to the system of $3N$ linear equations

$$A^n U^{n+1} = F^n \tag{11}$$

for unknown discrete sets $U^{n+1} = (\eta_1^{n+1}, \dots, \eta_N^{n+1}, u_1^{n+1}, \dots, u_N^{n+1}, \alpha_1^{n+1}, \dots, \alpha_N^{n+1})$. Here A^n is the matrix of the system, F^n is the right-hand side vector, and superscript n specifies a mesh point on the time axis. To solve the resulting linear system (11) with a sparse banded matrix, we should propose an effective numerical algorithm based on an appropriate iterative method. Following the basic principle, we split up the matrix of the system into three components:

$$A = T + T_l + T_u, \tag{12}$$

where T is a tridiagonal matrix, T_l and T_u are semidiagonal matrices with non-zero elements situated below and above the main diagonal, respectively. The new estimate of the solution at each time step is then produced by the following iterative procedure:

$$TU^{(k+1)} = F - T_l U^{(k)} - T_u U^{(k)} \tag{13}$$

The solution at the previous time step serves as inital guess ($k = 0$) in the iteration process. Here the superscript n for the time step is omitted and replaced by the index (k) denoting the iteration step. The solutions of the systems (13) at each successive iteration $k+1$ is obtained by use of the Gaussian Elimination method. We may speed up the iteration algorithm by introducing the relaxation parameter. But even in this primitive form the convergence rate is reasonably high, where the given tolerance of 10^{-6} is achieved within 15 iterations.

4. WAVE TRANSFORMATION AND BOTTOM DISPLACEMENT

The surface waves interact with the seafloor and cause the deflection of the elastic sheet lying on the viscoelastic foundation. The wave trasnformation depends on different values of the seafloor parameters: mass, rigidity, stiffness and damping coefficients. Figure 2 shows the wave profiles of the free surface and the elastic bottom at a given moment of time. It is observed that more flexible elastic foundation causes greater wave attenuation. With decrease in stiffness, the surface wave becomes shorter and the wave propagation rate becomes slower. The maximum magnitude of the bottom deflections, corresponding to the smallest possible stiffness parameter, makes about 10 percent of the wave elevation on the free surface.

Figure 3 illustrates the effect of viscous damping coefficient by showing the vibrations of the free surface and the seafloor at two wave gauges located at different distances from the wavemaker. Figure 3 also demonstrates the transformation of the wave due to the wave interaction with the seafloor by comparing the free surface elevations above the deformable seafloor and the flat rigid bottom. As surface wave penetrates deeper into the flow domain, the attenuation of the wave becomes stronger and the higher viscous damping causes larger dissipation of the wave.

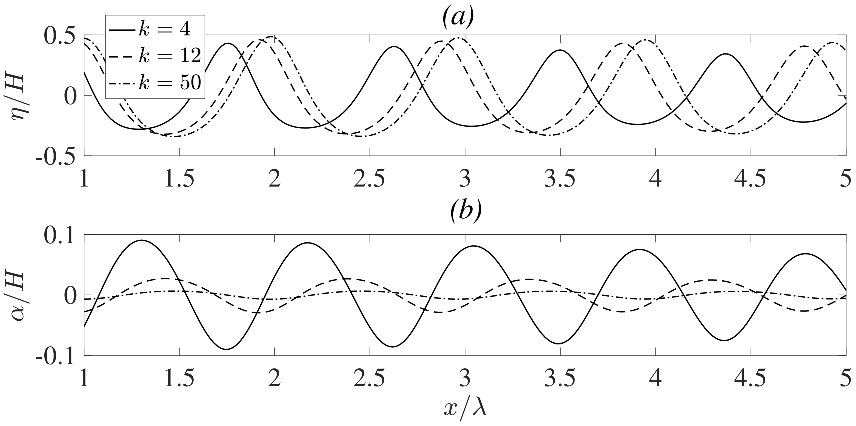

Figure 2: Snapshots of (a) free surface elevation and (b) bottom displacement at time $t = 100$ for cnoidal wave ($\lambda = 10$, $H = 0.25$) interaction with the elastic sheet ($m = 0.1$, $D = 0.1$) lying on the viscoelastic foundation ($b = 0.1$) of different stiffness k.

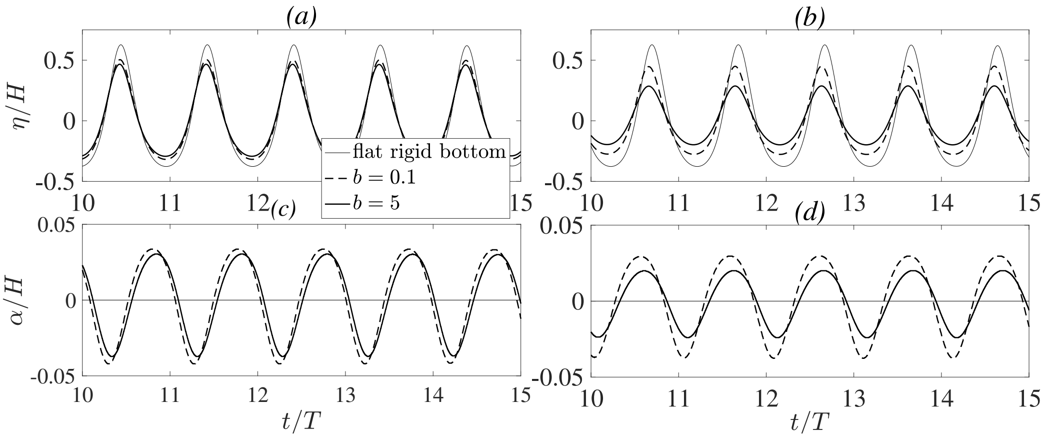

Figure 3: Time series of (a, b) free surface elevation and (c, d) bottom displacement at the wave gauges located (a, c) one wavelength and (b, d) five wavelengths from the wavemaker for the cnoidal wave ($\lambda = 12$, $H = 0.25$) interaction with elastic sheet ($m = 0.1$, $D = 0.1$) lying on the viscoelastic foundation ($k = 10$) with different damping coefficient b.

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