Ice response to the vertical inertial motion of an underwater circular cylinder

H. Xiong[a], **B. Y. Ni**[a], **A. A. Korobkin**[b]
a. College of Shipbuilding Engineering, Harbin Engineering University, Harbin, China
b. School of Mathematics, University of East Anglia, United Kingdom
Email: a.korobkin@uea.ac.uk

1 Introduction

Ice-breaking methods becomes even more important with developments of polar regions. It is known that flexural-gravity waves generated by a body moving under an ice cover can lead to significant ice deformation and even its damage. This was first shown theoretically by Kheisin (1967) and later experimentally by Kozin and Onishchuk (1994). However, the research on this topic was mainly foucued on ice deflection caused by horizontal motion of a underwater body at a constant speed. Small periodic perturbations of the body speed were studied by Stepanyants and Sturova (2021). Large-amplitude oscillations of a submerged circular cylinder under an ice cover were investigated by Li et al. (2017) assuming small deflections of the ice. The effects of free motions of underwater bodies on ice response have not been well studied. Ni et al. (2023) performed experiments with a buoyant sphere free-rising towards the ice cover from a certain depth. This research was focused on the collision between the sphere and the ice, and the following cracking the ice cover. In the present paper, we investigate the problem of a circular cylinder moving inertially towards a floating elastic plate from below.

2 Formulation of the problem

The ice response to the vertical inertial motion of an underwater circular cylinder is studied. The ice sheet floating on the water surface is modeled as a thin elastic plate of constant thickness h_i, density ρ_i, and rigidity $D = Eh_i^3/[12(1-\nu^2)]$, where E is Young's modulus of the ice and ν is Poisson's ratio. The liquid under the ice is inviscid, incompressible and of infinite depth. The liquid flow under the ice and the ice deflection are described in the Cartesian coordinate system $Ox'y'$, see figure 1. A prime stands for dimensional variables. A rigid cylinder of radius a starts moving vertically towards the ice cover at $t' = 0$ with the initial speed V. The motion of the cylinder is governed by its inertia, gravity, hydrodynamic force, deflection of the ice cover and the initial velocity. The liquid flow caused by the cylinder motion and ice deflection is assumed two-dimensional and potential.

In general, this problem is coupled because the ice deflection, hydrodynamic loads and the motion of the cylinder should be determined at the same time. The problem is formulated in dimensionless variables, where the radius of the cylinder a is taken as the length scale, a/V as the time scale, Va as the scale of the velocity potential and $\rho_w V^2$ as the scale of the hydrodynamic pressure, where ρ_w is the water density. The deflection scale $w_{sc} = \rho_w V^2 a^4/D$ is obtained by equating the order of the hydrodynamic pressure acting on the plate to the order of the bending term in the Kirchhoff equation of thin elastic plate. The dimensionless variables are denoted by the same symbols without a prime.

The dimensionless deflection of the elastic plate $w(x,t)$ is described by the equation

$$\chi\varepsilon\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = p(x, \varepsilon w(x,t), t) \quad (-\infty < x < \infty,\, t > 0), \tag{1}$$

where $\chi = \rho_i h_i/\rho_w a$, and $\varepsilon = w_{sc}/a$ is a small parameter of the problem. The plate equation (1) is solved subject to the far-field condition and the initial conditions,

$$w(x,t) \to 0 \quad (|x| \to \infty), \quad w(x,0) = 0, \quad w_t(x,0) = 0. \tag{2}$$

The elastic strains take their maximum values on the upper and lower surface of the ice sheet,

$$\epsilon(x,t) = \mp\frac{1}{2}\frac{h_i}{a}\varepsilon\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

Strains greater than the so-called yield strain of the ice, ϵ_{cr}, are assumed to lead to ice fracture. Here we take $\epsilon_{cr} = 8\times 10^{-5}$, see Brocklehurst et al. (2010) and discussions there.

The dimensionless hydrodynamic pressure $p(x,y,t)$ in the flow region, $\Omega(t) = \{x,y|y < \varepsilon w(x,t), \rho > 1\}$, is given by the nonlinear Bernoulli equation,

$$p(x,y,t) = -\varphi_t - \frac{1}{2}(\nabla\varphi)^2 - \mathrm{Fr}^{-2}y, \tag{4}$$

where $\mathrm{Fr} = V/\sqrt{ga}$ is the Froude number, g is gravitational acceleration, and $\varphi(x,y,t)$ is the velocity potential of the flow region, which satisfies the Laplace equation in the flow region,

$$\nabla^2\varphi = 0 \ \ ((x,y)\in\Omega(t)\,), \tag{5}$$

and the boundary conditions,

$$\varphi_\rho = -\dot{h}\sin\alpha \ \ (\rho = 1, 0\le\alpha\le 2\pi), \ \ \varphi_y = \varepsilon\left(w_x\varphi_x + w_t\right) \ \ (y = \varepsilon w(x,t)), \tag{6}$$

where an overdot stands for time derivative, and decays at infinity,

$$\varphi \to 0(x^2+y^2\to\infty). \tag{7}$$

The vertical motion of the cylinder is governed by Newtons second law,

$$M\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = \frac{1}{\pi}\int_0^{2\pi} p(1,\alpha,t)\sin\alpha\,\mathrm{d}\alpha + M\mathrm{Fr}^{-2}, \tag{8}$$

where M is the ratio of the mass of the cylinder M' to the mass of the water in the volume of the cylinder, $\pi a^2\rho_w$, per unit width, $M = M'/(\pi a^2\rho_w)$. The pressure $p(1,\alpha,t)$ in (8) is in the local moving polar coordinates ρ, α, where the surface of the cylinder is at $\rho = 1$, see figure 1. Equation (8) should be integrated in time subject to the initial conditions,

$$h(0) = h_0, \dot{h}(0) = -1, h_0 = h_0'/a \tag{9}$$

For $\varepsilon \ll 1$ the condition on the ice/water interface and the right-hand side in the plate equation (1) can be approximately simplified neglecting terms of order $\mathrm{O}(\varepsilon^2)$ and higher,

$$\varphi_y(x,0,t) = \varepsilon\left(w_t + [w(x,t)\varphi_x(x,0,t)]_x\right) + \mathrm{O}(\varepsilon^2)(y=0). \tag{10}$$

$$p(x,\varepsilon w(x,t),t) = -\varphi_t(x,0,t) - \frac{1}{2}\varphi_x^2(x,0,t) - Fr^{-2}\varepsilon w + \mathrm{O}(\varepsilon^2). \tag{11}$$

Note that the derivative φ_y on the ice/water interface is of order of $\mathrm{O}(\varepsilon)$ and the condition at this interface can be imposed on $y = 0$ with the order of $\mathrm{O}(\varepsilon^2)$.

3 Solution of the problem

The boundary value problem (5)-(7), where the terms of the order of $\mathrm{O}(\varepsilon^2)$ are neglected, is solved using the conformal mapping,

$$z/\mu = i + 2/(\zeta + i), \tag{12}$$

of a ring $R(t) < |\zeta| < 1$ in the complex ζ-plane, $\zeta = -ire^{i\theta}$, $-\pi < \theta < \pi$, onto the flow region $\Omega(t)$ in the complex z-plane, $z = x+iy$, see Wang (2004). Here $\mu = \sqrt{h^2-1}$ and $R = h(t) - \mu(t)$. Equation (10) defines the relations $x = x(r,\theta,t)$, $y = y(r,\theta,t)$, $\alpha = \alpha(r,\theta,t)$ and $\rho = \rho(r,\theta,t)$ between the coordinates in the physical plane and the polar coordinates r,θ in the plane of the conformal mapping, see figure 2. We neglect the second nonlinear term in the boundary condition (10) assuming it is small. This assumptions should be checked a posteriori.

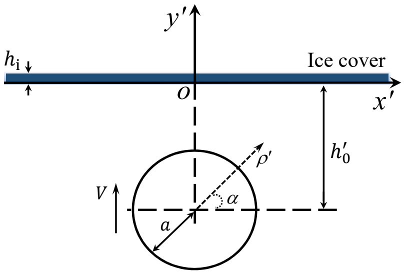

Figure 1: Sketch of the problem and notations

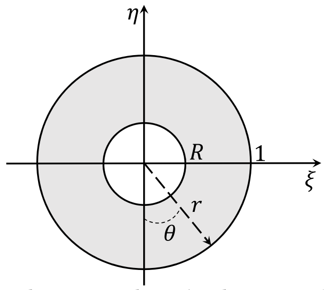

Figure 2: The complex ζ-plane and notations

The velocity potential in the ζ-plane, $\phi(r,\theta,t)=\varphi(x(r,\theta,t),y(r,\theta,t),t)$ can be determined if the velocity of the ice deflection, $w_t(x,t)$, and the function $h(t)$ are known. The problem (5)-(6) has a solution only if $\int_{-\infty}^{\infty} w(x,t)\mathrm{d}x{=}0$ at any time instant. Note that the deflection $w(x,t)$ is an even function of x. The deflection $w(x,t)$ is sought in the form

$$w(x(1,\theta,t),t)=(1-\cos\theta)\sum_{n=1}^{\infty} W_n(t)\cos(n\theta), \tag{13}$$

where $r=1$ corresponds to the ice/water interface and $\theta\to 0$ as $|x|\to\infty$. The function (13) is even, decays at infinity and satisfies the solvability condition.

Differentiating (13) with respect to time t, we obtain

$$w_t(x,t)=(1-\cos\theta)\sum_{n=1}^{\infty} Q_n\cos(n\theta)=\sum_{n=1}^{\infty}\tilde{Q}_n[1-\cos(n\theta)], \tag{14}$$

where $Q_n=\dot{W}_n+qW_n^{(t)}$, $q=-\dot{h}h\big/(2\mu^2)$, $W_1^{(t)}=W_2-2W_1$, $W_n^{(t)}=nW_{n+1}-nW_{n-1}-2W_n (n\geq 2)$, $\tilde{Q}_1=-Q_1+Q_2/2$, and $\tilde{Q}_n=Q_{n-1}/2-Q_n+Q_{n+1}/2$ $(n\geq 2)$. The velocity potential $\phi(r,\theta,t)$ satisfies the Laplace equation in the ring and the boundary conditions

$$\phi_r=2\frac{\dot{h}\mu}{R}\sum_{n=1}^{\infty} nR^n\cos(n\theta)\ \ (r=R),\ \phi_r=\mu\varepsilon\sum_{n=1}^{\infty} Q_n\cos(n\theta)\ \ (r=1) \tag{15}$$

The potential at $r=1$, which corresponds to the ice/water interface, and at $r=R$, which corresponds to the surface of moving cylinder, are given by

$$\phi(1,\theta,t)=\sum_{n=1}^{\infty}\left(4\dot{h}\mu\varphi_n-\mu\varepsilon\frac{1+2\varphi_n}{n}Q_n\right)(1-\cos(n\theta)),$$

$$\phi(R,\theta,t)=\sum_{n=1}^{\infty}\mu\left(4\dot{h}\varphi_n-\varepsilon\frac{1+2\varphi_n}{n}Q_n\right)+\sum_{n=1}^{\infty}2\mu\left(\frac{\varepsilon\varphi_nQ_n}{nR^n}-\dot{h}\,(1+2\varphi_n)\,R^n\right)\cos(n\theta) \tag{16}$$

where $\varphi_n=R^{2n}/(1-R^{2n})$ and the condition (7) was used. The dimensionless hydrodynamic pressure along the surface of the cylinder, $p(1,\alpha,t)$ in (8), is obtained through using (16) and (4).

$$\ddot{h}\,(M+M_a)=\dot{h}f_1+\dot{h}^2f_2+\sum_{n=1}^{\infty}4\mu^2\varepsilon\varphi_n\ddot{W}_n+F_0, \tag{17}$$

where

$$M_a=\sum_{n=1}^{\infty}\left(4\mu^2nR^{2n}\,(1+2\varphi_n)+2h\varepsilon\varphi_nW_n^{(t)}\right), f_1=\sum_{n=1}^{\infty}\left(12h\varepsilon\varphi_n\dot{W}_n+2\frac{h}{\mu}\varepsilon^2\frac{{\varphi_n}^2}{R^{2n}}\dot{W}_nW_n^{(t)}-2h\varepsilon\varphi_n\dot{W}_n^{(t)}\right)$$

$$f_2=\sum_{n=1}^{\infty}\left\{-12hn\,(1+2\varphi_n)\,R^{2n}+2\mu n^2\left(8{\varphi_n}^2+\left(1+2\varphi_n-4{\varphi_n}^2\right)R^{2n}\right)-\frac{h^2\varepsilon^2}{2\mu^2}\frac{{\varphi_n}^2}{R^{2n}}\left(W_n^{(t)}\right)^2-2\varepsilon\frac{2\mu^2+1}{\mu^2}\varphi_nW_n^{(t)}\right\}$$

$$F_0=(M-1)\,\mathrm{Fr}^{-2}-\sum_{n=1}^{\infty}2\mu\varepsilon^2\frac{{\varphi_n}^2}{R^{2n}}\left(\dot{W}_n\right)^2$$

Note that f_1 and F_0 depend on the time derivatives of the unknown coefficients in the series (13).

It is convenient to introduce a new unknown function $u(x,t)$ and decompose equation (1) as

$$\chi\varepsilon w_t + \varphi(x,0,t) = u(x,t) \tag{18}$$

$$u_t + w_{xxxx} + Fr^{-2}\varepsilon w = -\frac{1}{2}(\varphi_x)^2(x,0,t) \tag{19}$$

where $u(x,t)$ is sought in the form $u(x(1,\theta,t),t) = (1-\cos\theta)\sum_{n=0}^{\infty} U_n\cos(n\theta) = \sum_{n=1}^{\infty}\tilde{U}_n[1-\cos(n\theta)]$, which is suggested by (14) and (16). The right-hand side in (19) can be presented as $(\varphi_x)^2 = (1-\cos\theta)\sum_{n=0}^{\infty} V_n\cos(n\theta)$, where V_n is made up with $\dot{W}_n$ and W_n. Then substituting the obtained terms in (18) and (19), and eliminating the terms of , the infinite system of ordinary differential equations can be obtained,

$$(\chi\varepsilon\mathbb{B} - \mu\varepsilon\mathbb{D})\,\dot{\mathbf{W}} = -q\mathbb{C}\mathbf{W} - 4\dot{h}\mu\boldsymbol{\varphi} + \mathbb{B}\mathbf{U} + U_0\mathbf{e}_1 \tag{20}$$

$$\dot{U}_0 = \frac{1}{2}V_0 + 2qU_0,\;\; \dot{\mathbf{U}} = -q\mathbb{C}\mathbf{U} + 2qU_0\mathbf{e}_1 - \left(\frac{1}{\mu^4}\mathbb{A}^2 + Fr^{-2}\varepsilon\mathbf{I}\right)\mathbf{W} + \frac{1}{2}\mathbf{V} \tag{21}$$

where $\dot{\mathbf{W}} = (\mathrm{d}W_1/\mathrm{d}t, \mathrm{d}W_2/\mathrm{d}t, \cdots)^T$, $\mathbf{W} = (W_1, W_2, W_3\cdots)^T$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3\cdots)^T$, $\mathbf{e}_1 = (1,0,0,....)^T$. $\mathbb{B}$, $\mathbb{C}$ are three-diagonal symmetric matrix with the elements $B_{nn} = -1$, $B_{n,n-1} = B_{n,n+1} = \frac{1}{2}$, and $C_{11} = -2$, $C_{12} = 1$, $C_{nn} = -2$, $C_{n,n-1} = -n$, $C_{n,n+1} = n$ for $n \geq 2$. $\mathbb{D}$ is diagonal matrix with elements $(1+2\varphi_n)/n$. $\mathbb{A}$ is a constant five-diagonal matrix which is related to the second derivative in x of the series (13) and $\mathbf{I}$ is a unit matrix.

The system (20), (21) is integrated in time numerically subject to zero initial conditions

$$W_n(0) = 0,\;\; \mathbb{B}\mathbf{U} + U_0\mathbf{e}_1 = 4\dot{h}\mu\varphi,\;\; t = 0, \tag{22}$$

and equation (17).

The numerical results are presented in terms of the ice deflections and strains caused by the inertial motion of the cylinder. Calculations are performed for the following reference values of the parameters, ρ_i = 917 kg/m^3, E= 4.2 × 10^9 N/m^2, ν= 0.3, h_i= 0.2 m, g= 9.81m/s^2, ρ_w= 1000kg/m^3, V= 1m/s, h_0= 4m, and mass ratio $M = 2, 6, 10$. The system (20)-(21) is truncated to N terms in the series (13) for the deflection and (16) for the potential. Then the system (17), (20) and (21) of ordinary differential equations is integrated in time with the initial condition (22). It is numerically solved by fourth order Runge-Kutta method with dimensionless time step Δt= 10^{-5}. The convergence analysis of and with respect to the number of retained terms in (13) and (16) was investigated. Numerical results and the convergence tests will be presented at the workshop.

4 Acknowledgement

This work is supported by the National Natural Science Foundation of China (Nos.52192693, 52192690, 52371270, and U20A20327), to which the authors are most grateful.

5 References

Brocklehurst P., Korobkin A., Parau E., 2010. Interaction of hydro-elastic waves with a vertical wall, Journal of Engineering Mathematics, 68, 215-231.

Kheisin D. E., 1967. Ice-Field Dynamics (in Russian), Gidrometeoizdat, Leningrad.

Kozin V.M., Onishchuk A.V., 1994. Model investigations of wave formation in solid ice cover from the motion of a submarine. Journal of Applied Mechanics and Technical Physics. 35, 235-238.

Li Z. F., Shi Y. Y., Wu G. X., 2017. Large amplitude motions of a submerged circular cylinder in water with an ice cover. European Journal of Mechanics - B/Fluids, 65, 141-159.

Ni B. Y., Tan H., Di S.C., Zhang C.X., Li Z.Y., Huang L.F, Xue Y.Z., 2023. When does a light sphere break ice plate most by using its net buoyance? Journal of Marine Science and Engineering, 11(2), 289.

Stepanyants Y, Sturova I.V., 2021. Waves on a compressed floating ice plate caused by motion of a dipole in water. Journal of Fluid Mechanics, 907.

Wang X.Q., 2004. Interaction of two circular cylinders in inviscid fluid. Physics of Fluids, 16(12)