Normal mode method in a problem of hydroelastic waves in a frozen channel with an arbitrary thickness of ice

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1 Introduction and formulation of the problem

The problem of flexural-gravity wave propagation in the ice covers has been actively investigated for the last decades. The main part of the flexural-gravity wave research was carried out for ice sheets of infinite extent. However, majority of the ice tanks, where scientific and technical experiments with ice cover are conducted, have finite dimensions and rectangular cross-sections. The model of linear elastic plate is still the main one, although viscoelastic and poroelastic models of the ice cover are also considered by many authors. Nonlinear models, models that take into account ice compression and the presence of cracks in ice sheets, also have been actively developed (see, i.e., [1]–[3]). The problems of propagation of hydroelastic waves in a channel were studied by the authors of the article in different formulations and for different ice conditions in [4]–[5]. In this work, we shall study a problem of hydroelastic waves propagating along a frozen channel with variable ice thickness. The main attention is given to the effect of variable thickness on characteristics of periodic waves. The developed method of the solution can be used for approximation of any form of ice thickness across the channel.

The problem is studied within the linear theory of hydroelasticity. Deflections of the ice plate are described by the vertical displacement of the midsurface of the plate, $\hat{z} = \hat{w}(\hat{x}, \hat{y}, \hat{t})$. The mathematical formulation of the problem is coupled and consists of elastic and hydrodynamic parts. The elastic part is related to the determination of the ice deflections. The second part of the problem is related to the description of the hydrodynamics in the channel and determination of the flow potential $\hat{\varphi}(\hat{x}, \hat{y}, \hat{z}, \hat{t})$. The ice deflection $\hat{w}(\hat{x}, \hat{y}, \hat{t})$ satisfies the equation of a thin elastic plate with given constant density ρ_i and with variable thickness $\hat{h}_i(\hat{y})$ across the channel. The ice thickness does not change in $\hat{x}$ direction. We consider the ice frozen to the channel walls, which is modelled by clamped conditions at $\hat{y} = \pm b$. The potential $\hat{\varphi}(\hat{x}, \hat{y}, \hat{z}, \hat{t})$ satisfies the Laplace equation in the channel domain and boundary conditions of impermeability at rigid boundaries of the channel (walls and bottom) and linearized kinematic condition at the ice/liquid interface. Unknown functions $\hat{w}$ and $\hat{\varphi}$ are sought in the form of periodic hydroelastic waves propagating along the channel in the positive $\hat{x}$-direction with constant amplitude A, wavenumber k and frequency ω

$$\hat{w}(\hat{x}, \hat{y}, \hat{t}) = ARe(F(\hat{y})e^{i(k\hat{x}-\omega\hat{t})}),$$

$$\hat{\varphi}(\hat{x}, \hat{y}, \hat{t}) = -ARe(i\omega\Phi(\hat{y}, \hat{z})e^{i(k\hat{x}-\omega\hat{t})}).$$

The problem is solved in dimensionless variables. Dimensional variables are marked with hat, nondimensional variables have standard notations. Dimensional wavenumber is denoted by k, nondimensional – by κ, $\kappa = kb$. Length scale is taken to be half the channel width b, the time scale is $1/\omega$, the amplitude A is the scale of the ice deflection, $\rho_\ell g A$ is the scale of the liquid pressure, ρ_ℓ is the liquid density, g is the accleration of gravity, $Ab\omega$ is the scale of the flow velocity potential, $h = H/b$ is the dimensionless channel depth. We introduce an average value of the ice thickness across the channel, h_{avg}, and use it as a scale of the ice thickness $\hat{h}_i(\hat{y})$. We will consider only solutions with positive and real k and real $\hat{F}(\hat{y})$.

In the dimensionless variables equation of the plate reads

$$\beta \Bigg[h_i^3(y)\left(\kappa^4 F - 2\kappa^2 F'' + F''''\right) + 6h_i^2(y)h_i'\left(-\kappa^2 F' + F'''\right) + \\ +6h_i(y)(h_i')^2\left(F'' - \mu\kappa^2 F\right) - \frac{\delta\gamma}{\beta}F\Bigg] = \gamma\Phi(y,0) - F, \qquad (1)$$

where $\beta = D_{avg}/[\rho_\ell g b^4]$, $D_{avg} = Eh_{avg}^3/[12(1-\mu^2)]$, E is the Young's modulus for ice, μ is the Poisson ratio for ice, $\gamma = b\omega^2/g$ and $\delta = h_{avg}\rho_i/[b\rho_\ell]$, prime stands for the derivative with respect to y. The rest of the equations of the problem in the dimensionless variables are

$$F = 0, \quad F_y = 0 \qquad (y = \pm 1), \qquad (2)$$

$$\Phi_{yy} + \Phi_{zz} = \kappa^2\Phi \qquad (-1 < y < 1, -h < z < 0), \qquad (3)$$

$$\Phi_y = 0 \quad (y = \pm 1), \qquad \Phi_z = 0 \quad (z = -h), \qquad \Phi_z = F \quad (z = 0). \qquad (4)$$

The solution of the problem (1) – (4) depends on the three dimensionless parameters β, γ and δ. We shall find dispersion relations $\omega(k)$ for some typical values of parameters of the problem.

2 Method of the solution and discussion

The solution of the equation (1) is sought in the form of an infinite series of eigenmodes of oscillations of an elastic beam with varying thickness

$$F(y) = \sum_{n=1}^{\infty} a_n \psi_n(y), \qquad (5)$$

These spectral functions are introduced for each case of varying thickness, however, in general, the algorithm for determining these functions is the same and consists of repeated steps. To determine the functions $\psi_n(y)$, the segment $[-1, 1]$ across the channel is divided into small segments $[b_k, b_{k+1}]$, where $1 = b_0 < b_1 < ... < b_{N_k-1} < b_{N_k} = 1$. On each of these segments, the ice thickness $h_i(y)$ is approximated by a linear function $h_i^{(k)}(y)$. On the segments with constant thickness the function $h_i^k(y)$ is equal to $h_{avg}^{(k)}$. The functions $\psi_n^{(k)}$ on these segments are solutions of the spectral problem describing the oscillations of an elastic beam with a constant thickness

$$\left(h^{(k)}{}_{avg}\right)^2 \psi_{n,yyyy}^{(k)} = \theta_n^4 \psi_n^{(k)}(y) \quad (b_k \le y \le b_{k+1}), \qquad (6)$$

$$\psi_n^{(k)}(y) = A_n^{(k)}\cos(\theta_n^* y) + B_n^{(k)}\sin(\theta_n^* y) + C_n^{(k)}\cosh(\theta_n^* y) + D_n^{(k)}\sinh(\theta_n^* y), \quad \theta_n^* = \frac{\theta_n}{\sqrt{h^{(k)}{}_{avg}}}.$$

The expansion coefficients $A_n^{(k)}, B_n^{(k)}, C_n^{(k)}$ and $D_n^{(k)}$, as well as the spectral parameter θ_n, which is the same for all solutions on each segment, are determined from the orthogonality condition for these modes and the boundary conditions at the boundaries of the segments. For the case with linearly varying ice thickness, the values of the thickness on the left, $h_{left}^{(k)}$, on the right, $h_{right}^{(k)}$, and the average value of the thickness on this segment, $h_{avg}^{(k)}$, are introduced. Then the ice thickness is described by the function

$$h_i(y) = \alpha_0^{(k)} + \alpha^{(k)} y, \quad h_i(b_k) = h_{left}^{(k)}, \quad h_i(b_{k+1}) = h_{right}^{(k)},$$

where $\alpha_0^{(k)} = h_{right}^{(k)} - \alpha^{(k)} b_{k+1}$, $\alpha^{(k)} = (h_{right}^{(k)} - h_{left}^{(k)})/(b_{k+1} - b_k)$. The most important parameter is $\alpha^{(k)}$ which describes the angle of inclination for the linear change in ice thickness at given linear segment. Functions $\psi_n^{(k)}$ are solutions of the spectral problem describing the oscillations of an elastic beam with a linearly varying thickness

$$h_i^2(y)\psi_{n,yyyy}^{(k)} + 6h_i(y)h_{i,y}\psi_{n,yyy}^{(k)} + 6(h_{i,y})^2\psi_{n,yy}^{(k)} = \theta_n^4 \psi(k)_n \quad (b_k \le y \le b_{k+1}), \qquad (7)$$

$$\psi_n^{(k)}(y) = \frac{1}{\xi}\left[A_n^{(k)} J_1(\eta_n^{(k)}\xi) + B_n^{(k)} Y_1(\eta_n^{(k)}\xi) + C_n^{(k)} I_1(\eta_n^{(k)}\xi) + D_n^{(k)} K_1(\eta_n^{(k)}\xi)\right] \quad n = 1, 2, 3...,$$

where $\eta_n^{(k)} = 2\theta_n/\alpha^{(k)}$, $\xi = \sqrt{\alpha_0^{(k)} + \alpha^{(k)} y}$ and J, Y, I, K are Bessel functions. When the solutions on each segment are determined, the resulting function $\psi_n(y)$ is introduced as piece-wised function equal to $\psi_n^{(k)}$ on each segment. At the boundaries of the segments, $y = b_k$, $k = 1, 2, ..., N_k - 1$, it is necessary that the continuity conditions of ice deflections, slope, bending moments and shear forces are satisfied. The last needed equation is the orthogonality condition of the modes $\psi_n(y)$ with a weight on each segment.

After determining the normal modes, the solution for the potential is sought in the form of a series of functions ϕ_n multiplied by the same coefficients a_n as in the series (5). The functions ϕ_n are found from the corresponding boundary problems in the channel cross-section and are expressed in terms of the modes ψ_n. Subsequently, these expansions are substituted into the plate equation, and the result is multiplied by a single mode ψ_n, leading to an infinite system of algebraic equations to find a_n and dispersion relations $\omega_n(k)$. The system is solved using a reduction method, which is similar to the approach outlined in [5].

Calculations of periodic hydroelastic waves and their characteristics were carried out for parameters of the problem corresponding to the experimental ice tank at the Sholem Aleichem Amur State University in Birobidzhan (see [2]): $H = 1$ m, $2b = 3$ m. The parameters of ice and liquid in the calculations were: $\rho_i = 917$ kg/m^3, $\rho_\ell = 1024$ kg/m^3, $\mu = 0.3$, $E = 4.2 \cdot 10^9$ Pa. The average ice thickness h_{avg} in the tank is equal to 0.0035 m in our calculations.

We keep the same average thickness h_{avg} in the calculations for different sets of parameters when describing the ice thickness of the ice cover across the channel. Algorithm of the solution will be demonstrated on one given case of variable thickness of ice cover (Figure 1a). A piecewise approximation of this thickness is shown in Figure 1b. The ice thickness consists of 3 parts: the left linear part, $-1 < y \le b_1$, the right linear part, $b_2 < y \le 1$, and in the middle part the ice thickness is constant and equal to $h_{avg}^{(2)}$. The average value of the thickness on the left and right segments is defined as $h_{avg}^{(1)}$ and $h_{avg}^{(3)}$. On the presented Figure, the left part corresponds to a large change in thickness on a short segment, the right part corresponds to an average change in thickness on a long segment. The linear change on these segments is described by the parameters $\alpha^{(k)}$.

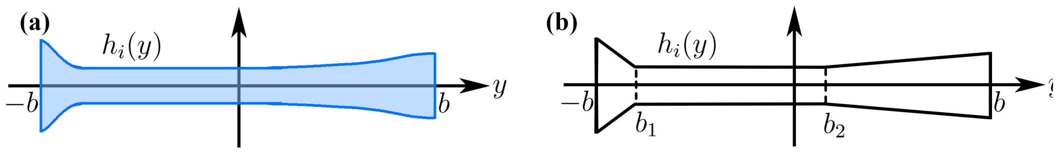

Figure 1: Shape of the ice thickness across the channel used in the calculations **(a)**. A piecewise approximation of this thickness **(b)**.

The phase speeds of the first two waves with the lowest frequency are shown in Figure 2(a). The black line represents the results for ice with constant thickness, while the colored lines depict the results for ice with variable thickness, as shown in Figure 1, for different sets of parameters. In Figure 2(b), profiles of a short wave with $k = 10\text{m}^{-1}$ and the lowest frequency are shown. Overall, an increase in variation of ice thickness (reduction of its average value in the central segment and increase of its average value in the lateral segments) leads to a significant change (reduction) in speeds and a shift of the location of maximum deflections. The provided calculation example demonstrates that hydroelastic waves in an ice cover with non-uniform ice thickness can be investigated based on the presented method. The characteristics of these waves strongly depend on the specific distribution of ice thickness in the channel. The considered variation in ice thickness over just three segments leads to a significant change in the characteristics of the periodic hydroelastic waves. More detailed results will be presented at the Workshop.

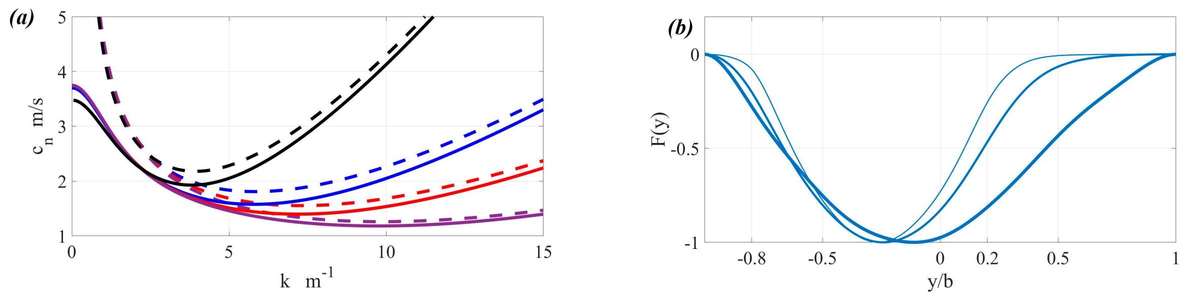

Figure 2: Phase speeds of the first two periodic hydroelastic waves with the lowest frequencies propagating in the channel **(a)**. The ice deflections of a short wave with the lowest frequency across the channel, $x = 0$ **(b)**.

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