Second order theory for interaction of flexible floating body with waves

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Introduction

The second order interactions of the floating rigid body with water waves have been widely studied in the past, and the consistent theoretical model was agreed within the community ([1],[2],[3],[4]). The case of flexible body has been studied seldomly and, to the author's knowledge, there is no consistent theoretical formulation which has been proposed in the literature. Here we try to do that.

Notations and coordinate systems

We introduce the compact matrix notations where $[\boldsymbol{A}]$ is used to denote the $n \times m$ matrix with the elements A_{ij} , $(i = 1, n\,; j = 1, m)$ and, any vector quantity $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$, is written as a single column matrix $\{\boldsymbol{a}\}$. At the same time, to each vector quantity, the skew symmetric matrix $[\boldsymbol{a}]$ is associated so that the scalar and vector product of two vectors $\{\boldsymbol{a}\}$ and $\{\boldsymbol{b}\}$ can be written as a classical matrix product: $\boldsymbol{a} \cdot \boldsymbol{b} = \{\boldsymbol{a}\}^T\{\boldsymbol{b}\}$, $\boldsymbol{a} \wedge \boldsymbol{b} = [\boldsymbol{a}]\{\boldsymbol{b}\}$.
When formulating the wave body interaction, several coordinate systems need to be defined. Most often, in addition to the inertial (earth fixed) coordinate system (x, y, z), the coordinate system fixed to the body (x', y', z') is introduced, as shown in Figure 1. The earth fixed coordinate system is usually chosen to have an arbitrary origin fixed in space, while the body fixed coordinate system is commonly chosen to be fixed at the body center of gravity:

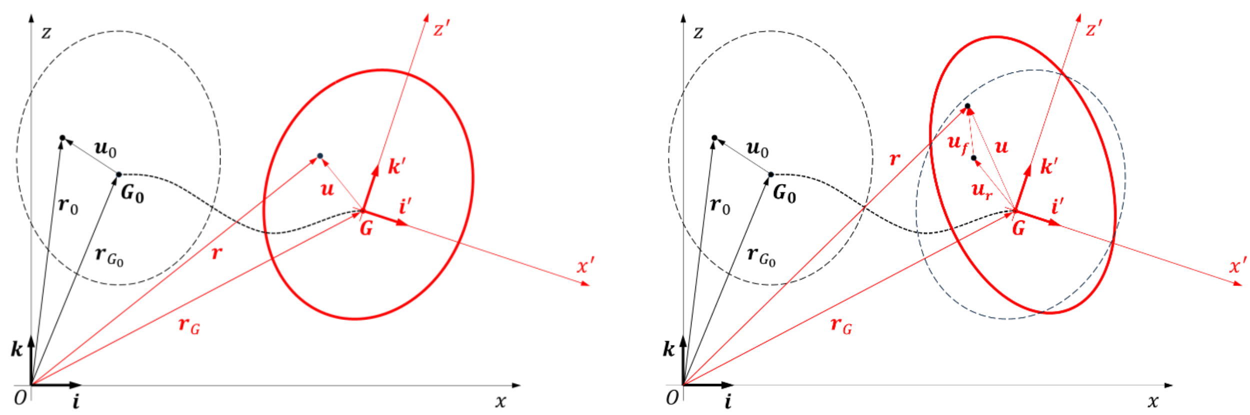

Figure 1: Rigid (left) and flexible (right) body motion and the different coordinate systems.

Any vector quantity $\boldsymbol{b}$ can be expressed in either earth fixed or body fixed coordinate system. The two sets of coordinates are related to each other by the transformation matrix $[\boldsymbol{A}]$, so that for any vector quantity $\{\boldsymbol{b}\}$ we can write $\{\boldsymbol{b}\} = [\boldsymbol{A}]\{\boldsymbol{b}'\}$ where it should be clearly understood that both $\{\boldsymbol{b}\}$ and $\{\boldsymbol{b}'\}$ represent the same vector in space. The transformation matrix $[\boldsymbol{A}]$ is the critical parameter in the analysis of the nonlinear body motion and the instantaneous body rotation vector $\{\boldsymbol{\Omega}'\}$ is related to the transformation matrix by $[\boldsymbol{\Omega}'] = [\boldsymbol{A}]^T[\dot{\boldsymbol{A}}]$. The transformation matrix can be chosen in many different ways (Euler angles, quaternions, Rodriguez formula …) and here the Euler angles with convention (321) are used and the description of the problem is made relative to the body fixed coordinate system.

Second order theory

When formulating the second order problem there are different technical issues to be considered and the following four are probably the most important: description of the body motions, dynamic equilibrium equation, evaluation of the loads and the solution of the Boundary Value Problems (BVP) for the velocity potentials. We consider each of them explicitly below.

Instantaneous position of the body

The motion of the flexible body is described by the following position, velocity, and acceleration vectors: (e.g. see [5])

$$\begin{aligned}
\{\boldsymbol{r}\} &= \{\boldsymbol{r}_G\} + [\boldsymbol{A}]\{\boldsymbol{u}'\} \\
\{\boldsymbol{v}\} &= \{\boldsymbol{v}_G\} + [\boldsymbol{A}][\boldsymbol{\Omega}']\{\boldsymbol{u}'\} + [\boldsymbol{A}]\{\dot{\boldsymbol{u}}_f'\} \\
\{\boldsymbol{a}\} &= \{\boldsymbol{a}_G\} + [\boldsymbol{A}][\boldsymbol{\Omega}'][\boldsymbol{\Omega}']\{\boldsymbol{u}'\} + [\boldsymbol{A}][\dot{\boldsymbol{\Omega}}']\{\boldsymbol{u}'\} + 2[\boldsymbol{A}][\boldsymbol{\Omega}']\{\dot{\boldsymbol{u}}_f'\} + [\boldsymbol{A}]\{\ddot{\boldsymbol{u}}_f'\}
\end{aligned} \qquad (1)$$

The local position vector $\{\boldsymbol{u}\}$ is decomposed in two parts $\{\boldsymbol{u}\} = \{\boldsymbol{u}_r\} + \{\boldsymbol{u}_f\} = [\boldsymbol{A}]\{\boldsymbol{u}_0\} + [\boldsymbol{A}]\{\boldsymbol{u}'_f\}$ so that the rigid body case is obtained by simply putting $\{\boldsymbol{u}'_f\} = 0$. It is very important to note that the body deformations are defined relative to the body fixed coordinate system, which is located in the body center of gravity, and which moves following the rigid body motions only. This is the so-called Floating Frame of Reference (FFR) formulation of the flexible body dynamics as described in Shabana [5]. In the general case, the body deformation vector $\{\boldsymbol{u}'_f\}$ can be arbitrarily large, but here it is assumed to be small and is represented as a sum of the N_f modal contributions:

$$\{\boldsymbol{u}'_f(\boldsymbol{u}_0, t)\} = \sum_{i=1}^{N_f} \chi_{fi}(t)\{\boldsymbol{h}'_{0fi}(\boldsymbol{u}_0)\} = [\boldsymbol{\hbar}'_{0f}]\{\boldsymbol{\chi}_f\} \tag{2}$$

where $\{\boldsymbol{h}'_{0fi}\}$ is the space dependent modal displacement and $\chi_{fi}(t)$ its time dependent amplitude.
When compared to rigid body, the flexible body case, requires very careful description of the body instantaneous position the body normal vector and the elementary volume, which are modified by body deformations. For that purpose, it is convenient to introduce the notion of the deformation gradient $[\boldsymbol{F}]$ which is defined relative to body fixed coordinate system:

$$[\boldsymbol{F}] = [\boldsymbol{I}] + [\nabla \boldsymbol{u}'_f] \quad , \quad [\nabla \boldsymbol{u}'_f] = \begin{bmatrix} \frac{\partial u'_{fx'}}{\partial x'} & \frac{\partial u'_{fx'}}{\partial y'} & \frac{\partial u'_{fx'}}{\partial z'} \\ \frac{\partial u'_{fy'}}{\partial x'} & \frac{\partial u'_{fy'}}{\partial y'} & \frac{\partial u'_{fy'}}{\partial z'} \\ \frac{\partial u'_{fz'}}{\partial x'} & \frac{\partial u'_{fz'}}{\partial y'} & \frac{\partial u'_{fz'}}{\partial z'} \end{bmatrix} \tag{3}$$

With these notations, the normal vector and the elementary volume can be written as:

$$\{\boldsymbol{n}'\}dS = \|\boldsymbol{F}\|([\boldsymbol{F}]^{-1})^T\{\boldsymbol{n}_0\} \quad , \quad dV = \|\boldsymbol{F}\|dV_0 \tag{4}$$

This means that the following expressions can be deduced up to order $O\left((\boldsymbol{u}'_f)^2\right)$:

$$\{\boldsymbol{n}'\}dS = \left(1 + \nabla \boldsymbol{u}'_f - [\nabla \boldsymbol{u}'_f]^T + [\mathbf{Q}]^T\right)\{\boldsymbol{n}_0\}dS_0 \quad , \quad dV = \left(1 + \nabla \boldsymbol{u}'_f + \mathrm{Tr}[\mathbf{Q}]\right)dV_0 \tag{5}$$

where $\mathrm{Tr}[\,\cdot\,]$ denotes the trace of the matrix, and the matrix $[\mathbf{Q}]^T$ is given by:

$$[\mathbf{Q}]^T = \left[\left\{\frac{\partial \boldsymbol{u}'_f}{\partial y'} \wedge \frac{\partial \boldsymbol{u}'_f}{\partial z'}\right\} \quad \left\{\frac{\partial \boldsymbol{u}'_f}{\partial z'} \wedge \frac{\partial \boldsymbol{u}'_f}{\partial x'}\right\} \quad \left\{\frac{\partial \boldsymbol{u}'_f}{\partial x'} \wedge \frac{\partial \boldsymbol{u}'_f}{\partial y'}\right\}\right] \tag{6}$$

Dynamic equilibrium equation

The nonlinear motion equation can be deduced in the following compact form (e.g. see [5]):

$$[\boldsymbol{\mathcal{M}}']\{\ddot{\boldsymbol{\xi}}'\} + \begin{Bmatrix} \{0\} \\ \{0\} \\ [\boldsymbol{K}_{ff}]\{\boldsymbol{\chi}_f\} \end{Bmatrix} = \{\boldsymbol{\mathcal{F}}'\} - \{\boldsymbol{Q}'\} \tag{7}$$

where $[\boldsymbol{\mathcal{M}}']$ is the inertia matrix, $[\boldsymbol{K}_{ff}]$ is the structural stiffness matrix for the flexible modes, $\{\boldsymbol{\mathcal{F}}'\}$ is the external loading vector and $\{\boldsymbol{Q}'_{vi}\}$ are the quadratic velocity inertia vectors.

$$[\boldsymbol{\mathcal{M}}'] = \begin{bmatrix} [\boldsymbol{m}] & [0] & [0] \\ [0] & [\boldsymbol{I}'_{\theta\theta}] & [\boldsymbol{I}'_{\theta f}] \\ [0] & [\boldsymbol{I}'_{\theta f}]^T & [\boldsymbol{I}'_{ff}] \end{bmatrix}, \quad \{\ddot{\boldsymbol{\xi}}'\} = \begin{Bmatrix} \{(\ddot{\boldsymbol{r}}_G)'\} \\ \{\dot{\boldsymbol{\Omega}}'\} \\ \{\ddot{\boldsymbol{\chi}}_f\} \end{Bmatrix}, \quad \{\boldsymbol{\mathcal{F}}'\} = \begin{Bmatrix} \{\boldsymbol{F}'\} \\ \{\boldsymbol{M}'\} \\ \{\boldsymbol{F}'_f\} \end{Bmatrix}, \quad \{\boldsymbol{Q}'\} = \begin{Bmatrix} \{\boldsymbol{Q}'_{vR}\} \\ \{\boldsymbol{Q}'_{v\theta}\} \\ \{\boldsymbol{Q}'_{vf}\} \end{Bmatrix} \tag{8}$$

External loads

In the present context, the total external loading $\{\boldsymbol{\mathcal{F}}'\}$ is the sum of the gravity $\{\boldsymbol{\mathcal{F}}^{g\prime}\}$ and the pressure contribution $\{\boldsymbol{\mathcal{F}}^{h'}\}$. It is important to note that the gravity force is independent of the body instantaneous position, and it always acts in the direction of the acceleration of gravity. On the other hand, the direction of the pressure forces is defined by the normal vector on the wetted body surface so that its direction changes in time. These two facts have important consequences on the description of the external loading in the respective coordinate systems (earth fixed and body fixed).
Here we introduce the compact notations, so that we can write for the generalized loading, relative to body fixed coordinate system, the following expressions:

$$\{\boldsymbol{\mathcal{F}}^{g'}\} = -g\iiint_V \begin{Bmatrix} [\boldsymbol{A}]^T \\ [\boldsymbol{u}'][\boldsymbol{A}]^T \\ \{\boldsymbol{h}'_{fi}\}[\boldsymbol{A}]^T \end{Bmatrix} \{\boldsymbol{k}\}\rho_m dV \quad , \quad \{\boldsymbol{\mathcal{F}}^{h'}\} = \iint_{S_B} P\{\mathbb{N}'\}dS \quad , \quad \{\mathbb{N}'\} = \begin{Bmatrix} \{\boldsymbol{n}'\} \\ [\boldsymbol{u}']\{\boldsymbol{n}'\} \\ \{\boldsymbol{h}'_{fi}\}\{\boldsymbol{n}'\} \end{Bmatrix} \tag{9}$$

Within the potential flow theory, which is adopted here, the external pressure P is given by the Bernoulli's equation and, for the sake of clarity, it is further decomposed into its dynamic and the hydrostatic part:

$$P = -\varrho\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + gz\right] = P^d - \varrho gz \tag{10}$$

Linearization

Within the linearization process, all the physical quantities, either scalar q' or vector $\{\boldsymbol{q}'\}$, are first developed into perturbation series with respect to small parameter ε (wave steepness in the present case) and we can formally write:

$$q' = q'_0 + \varepsilon q'^{(1)} + \varepsilon^2 q'^{(1)} + O(\varepsilon^3) \quad , \quad \{\boldsymbol{q}'\} = \{\boldsymbol{q}'_0\} + \varepsilon\{\boldsymbol{q}'^{(1)}\} + \varepsilon^2\{\boldsymbol{q}'^{(1)}\} + O(\varepsilon^3) \tag{11}$$

In parallel, the Taylor series expansion is used to express the quantities at their instantaneous position as a function of their values at rest. In the case of flexible body, in addition to the rigid body displacements, the body deformations need to be accounted for so that the Taylor series expansion, relative to the body fixed coordinate system, can be written as:

$$q' = \left(1 + \varepsilon\{\boldsymbol{u}'_f\}\nabla\right)q'_0 \quad , \quad \{\boldsymbol{q}'\} = \left(1 + \varepsilon\{\boldsymbol{u}'_f\}\nabla\right)\{\boldsymbol{q}'_0\} \quad , \quad \{\boldsymbol{u}'_f\}\nabla = u'_{fx'}\frac{\partial}{\partial x'} + u'_{fy'}\frac{\partial}{\partial y'} + u'_{fz'}\frac{\partial}{\partial z'} \tag{12}$$

One of the most important quantities, in the present context, is the transformation matrix which is decomposed as follows:

$$[\boldsymbol{A}] = [\,\mathbf{I}\,] + \varepsilon[\boldsymbol{A}^{(1)}] + \varepsilon^2[\boldsymbol{A}^{(2)}] + O(\varepsilon^3) \tag{13}$$

where $[\,\mathbf{I}\,]$ denotes the identity matrix (all terms zero except the diagonal elements equal to 1) and, whatever the convention which is used for the definition of the transformation matrix, we can write:

$$[\boldsymbol{A}^{(1)}] = [\boldsymbol{\theta}^{(1)}] \quad , \quad [\boldsymbol{A}^{(2)}] = [\boldsymbol{\theta}^{(2)}] - \frac{1}{2}([\boldsymbol{H}]_S + [\boldsymbol{H}]_{AS}) = [\boldsymbol{\theta}^{(2)}] - \frac{1}{2}[\boldsymbol{\mathcal{H}}] \tag{14}$$

where $[\boldsymbol{H}]_S$ is the symmetric matrix given by $[\boldsymbol{H}]_S = -[\boldsymbol{\theta}^{(1)}][\boldsymbol{\theta}^{(1)}]$ and $[\boldsymbol{H}]_{AS}$ is the skew symmetric matrix which depends on the convention which is used for the definition of $[\boldsymbol{A}]$.

Body motions/deformations

The main unknowns of the problem are the time dependent amplitudes of the different modes of motion, and we decompose them as follows:

$$\{\boldsymbol{\xi}'\} = \varepsilon\{\boldsymbol{\xi}'^{(1)}\} + \varepsilon^2\{\boldsymbol{\xi}'^{(2)}\} + O(\varepsilon^3) = \varepsilon\begin{Bmatrix} \{(\boldsymbol{r}_G)'^{(1)}\} \\ \{\boldsymbol{\Omega}'^{(1)}\} \\ \{\boldsymbol{\chi}_f^{(1)}\} \end{Bmatrix} + \varepsilon^2\begin{Bmatrix} \{(\boldsymbol{r}_G)'^{(2)}\} \\ \{\boldsymbol{\Omega}'^{(2)}\} \\ \{\boldsymbol{\chi}_f^{(2)}\} \end{Bmatrix} + O(\varepsilon^3) \tag{15}$$

It should also be noted that the rotational velocity vector $\{\boldsymbol{\Omega}'\}$ is usually represented as a function of the time derivatives of the instantaneous rotational angles (Euler angles here) and we can write:

$$\{\boldsymbol{\Omega}'^{(1)}\} = \{\dot{\boldsymbol{\theta}}^{(1)}\} \quad , \quad \{\boldsymbol{\Omega}'^{(2)}\} = \{\dot{\boldsymbol{\theta}}^{(2)}\} + [\mathbf{G}'^{(1)}]\{\dot{\boldsymbol{\theta}}^{(1)}\} \tag{16}$$

The matrix $[\mathbf{G}'^{(1)}]$ follows from the fact that the rotational velocity vector $\{\boldsymbol{\Omega}\}$ and the time derivatives of the instantaneous rotation angles are related to each other by $\{\boldsymbol{\Omega}\} = [\mathbf{G}]\{\dot{\boldsymbol{\theta}}\}$. The "conversion" matrix $[\mathbf{G}]$ depends on the convention which is used for the definition of the transformation matrix $[\boldsymbol{A}]$. With this in mind, any other local quantity at the body (local position, mode shape, local displacement, local velocity; normal vector, elementary volume …) can be deduced.

External loading

The perturbation series for the external loads is formally written as:

$$\{\boldsymbol{\mathcal{F}}'\} = \{\boldsymbol{\mathcal{F}}^{g'}\} + \{\boldsymbol{\mathcal{F}}^{h'}\} = \{\boldsymbol{\mathcal{F}}^{g'(0)}\} + \{\boldsymbol{\mathcal{F}}^{h'(0)}\} + \varepsilon\left(\{\boldsymbol{\mathcal{F}}^{g'(1)}\} + \{\boldsymbol{\mathcal{F}}^{h'(1)}\}\right) + \varepsilon^2\left(\{\boldsymbol{\mathcal{F}}^{g'(2)}\} + \{\boldsymbol{\mathcal{F}}^{h'(2)}\}\right) \tag{17}$$

The different components are obtained, after quite heavy developments, in the following form:

$$\{\boldsymbol{\mathcal{F}}^{g'(0)}\} = \quad -g\iiint_{V_0}\begin{Bmatrix} 1 \\ [\boldsymbol{u}_0] \\ \{\boldsymbol{h}'_{0fi}\}^T \end{Bmatrix}\{\boldsymbol{k}\}\rho_m dV_0 \tag{18}$$

$$\{\boldsymbol{\mathcal{F}}^{g'(1)}\} = \quad -g\iiint_{V_0}\begin{Bmatrix} [\boldsymbol{A}^{(1)}]^T + V'^{(1)} \\ [\boldsymbol{u}_0]\left([\boldsymbol{A}^{(1)}]^T + V'^{(1)}\right) + [\boldsymbol{u}'^{(1)}_f] \\ \{\boldsymbol{h}'_{0fi}\}^T\left([\boldsymbol{A}^{(1)}]^T + V'^{(1)}\right) + \left\{\boldsymbol{h}'^{(1)}_{fi}\right\}^T \end{Bmatrix}\{\boldsymbol{k}\}\rho_m dV_0 \tag{19}$$

$$\{\boldsymbol{\mathcal{F}}^{g'(2)}\} = \quad -g\iiint_{V_0}\begin{Bmatrix} [\boldsymbol{A}^{(2)}]^T + V'^{(2)} \\ [\boldsymbol{u}_0]\left([\boldsymbol{A}^{(2)}]^T + V'^{(2)}\right) + [\boldsymbol{u}'^{(1)}_f]\left([\boldsymbol{A}^{(1)}]^T + V'^{(1)}\right) + [\boldsymbol{u}'^{(2)}_f] \\ \{\boldsymbol{h}'_{0fi}\}^T\left([\boldsymbol{A}^{(2)}]^T + V'^{(2)}\right) + \left\{\boldsymbol{h}'^{(1)}_{fi}\right\}^T\left([\boldsymbol{A}^{(1)}]^T + V'^{(1)}\right) + \left\{\boldsymbol{h}'^{(2)}_{fi}\right\}^T \end{Bmatrix}\{\boldsymbol{k}\}\rho_m dV_0 \tag{20}$$

$$\{\boldsymbol{\mathcal{F}}^{h'(0)}\} = \quad -\varrho g\iint_{S_{B_0}} z^{(0)}\{\mathbb{N}^{(0)}\}dS \tag{21}$$

$$\{\boldsymbol{\mathcal{F}}^{h'(1)}\} = \quad \iint_{S_{B_0}}\left(P^{(1)} - \varrho g z^{(1)}\right)\{\mathbb{N}^{(0)}\}dS \tag{22}$$

$$\{\boldsymbol{\mathcal{F}}^{h'(2)}\} = \quad \iint_{S_{B_0}}\left(P^{(2)} - \varrho g z^{(2)}\right)\{\mathbb{N}^{(0)}\}dS + \frac{1}{2}\varrho g\int_{C_{B_0}}\left(\Xi^{(1)} - z^{(1)}\right)^2\frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma}dC \tag{23}$$

where the compact normal vector is given by:

$$\{\mathbb{N}'^{(0)}\} = \begin{Bmatrix} \{\boldsymbol{n}_0\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}_0\} \\ \{\boldsymbol{h}'_{0fi}\}^T\{\boldsymbol{n}_0\} \end{Bmatrix}, \quad \{\mathbb{N}'^{(1)}\} = \begin{Bmatrix} \{\boldsymbol{n}'^{(1)}\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}'^{(1)}\} + \left[\boldsymbol{u}'^{(1)}_f\right]\{\boldsymbol{n}_0\} \\ \{\boldsymbol{h}'_{0fi}\}^T\{\boldsymbol{n}'^{(1)}\} + \left\{\boldsymbol{h}'^{(1)}_{fi}\right\}^T\{\boldsymbol{n}_0\} \end{Bmatrix}, \quad \{\mathbb{N}'^{(2)}\} = \begin{Bmatrix} \{\boldsymbol{n}'^{(2)}\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}'^{(2)}\} + \left[\boldsymbol{u}'^{(1)}_f\right]\{\boldsymbol{n}'^{(1)}\} + \left[\boldsymbol{u}'^{(2)}_f\right]\{\boldsymbol{n}_0\} \\ \{\boldsymbol{h}'_{0fi}\}^T\{\boldsymbol{n}'^{(2)}\} + \left\{\boldsymbol{h}'^{(1)}_{fi}\right\}^T\{\boldsymbol{n}'^{(1)}\} + \left\{\boldsymbol{h}'^{(2)}_{fi}\right\}^T\{\boldsymbol{n}_0\} \end{Bmatrix}$$

The line integral over the mean waterline C_{B_0} which occurs in (23), follows after careful integration over the instantaneous wetted body surface which should be decomposed into its mean and the time varying part, as shown in Figure 2

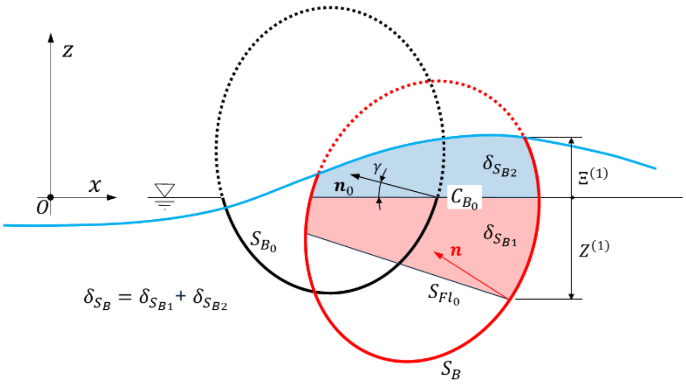

Figure 2: Instantaneous wetted body surface and its separation into the different integration surfaces.

Boundary Value Problems for velocity potentials

Due to lack of space, the BVP's for the different velocity potentials are not explicitly discussed here but will be at the workshop. We just mention that they remain very similar to rigid body case provided that the body boundary condition should account for the body velocity, and the change of the normal vector, which are induced by the body deformations.

References

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