A High-Order Finite Difference Incompressible Navier-Stokes model for Water Waves and Wave-Structure Interaction

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We propose a new high-order[1] finite difference numerical model for the simulation of nonlinear water waves and wave-structure interaction with fixed structures using the Navier-Stokes equations. The complete formulation is described in three spatial dimensions (3D) and preliminary validation results are here presented for two spatial dimensions (2D). A spatially fixed computational domain is defined through introducing a σ-coordinate that transform the Navier–Stokes equations along the vertical dimension from the sea bed to the still water level. Numerical experiments highlights both the correctness of the solution, the high-order convergence property that is attractive for efficient solutions as well as the classical benchmark problem due to Beji & Battjes (1994) where experimental measurements are available and hence serve as validation of the new high-order numerical scheme.

Governing equations

Consider a free surface Eulerian formulation of the Navier-Stokes equations. A Cartesian coordinate system is adopted with the xy-plane located at the still-water level at depth $h(x, y)$ and the z-axis pointing upwards. Assume that the water fluid density ρ is constant, i.e. the fluid is incompressible, which is expressed in terms of the mass continuity equation

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \Omega, \tag{1}$$

which describe that the divergence of the velocity field is zero everywhere in the fluid domain Ω. The divergence operator is defined as $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $\mathbf{u} = (u, v, w)$ is the velocity field vector.

The conservation of momentum is expressed by the Navier-Stokes equations

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F} - \rho(\mathbf{u} \cdot \nabla)\mathbf{u}, \quad t > 0, \quad \mathbf{x} \in \Omega, \tag{2}$$

where μ is the dynamic viscosity assumed to be constant $\mu = 1.3059$ mPA·s (at 10° C), p is the pressure and $\mathbf{F}$ is the net force vector acting on the fluid volume, assumed to be of the form $\mathbf{F} = \rho \mathbf{g}$ with $\mathbf{g} = (0, 0, -g_z)$ accounting for the gravitational effects in the vertical direction also assumed to be constant $g_z = 9.81$ m^2/s. Remark, if $\mu = 0$ the model is inviscid and hence an Euler model.

The evolution of the free water surface described in terms of the surface elvation $\eta(x, y, t)$ and is governed by the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = w - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y}, \quad t > 0, \quad z = \eta(x, y, t). \tag{3}$$

[1]We define a high-order numerical scheme as a scheme that has an order of accuracy of at least three.

There is no governing equation for the pressure. To relate the velocity field to the pressure it is assumes that the divergence and rate of change operators can be commuted. Thereby, a geometric conservation law for mass continuity can be stated in the form

$$\rho\frac{\partial(\nabla\cdot\mathbf{u})}{\partial t} = -\nabla^2 p + \mu\nabla\cdot(\nabla^2\mathbf{u}) - \rho\nabla\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}, \quad t>0, \quad \mathbf{x}\in\Omega. \tag{4}$$

Through developing a numerical scheme, the aim is to enforce that $\nabla\cdot\mathbf{u}=0$ at all discrete times, implying that the rate of change of the divergence of the velocity field is zero.

For the modelling of free surface waves, following Li & Flemming (2001) we assume that the reference pressure is $p = 0$ Pa at the free surface level. Also, we split the pressure into a dynamic p_D and a static contribution p_S such that

$$p = p_D + p_S, \quad p_S = \rho g_z(\eta - z). \tag{5}$$

For the design of wave propagation problems, we introduce a σ-transformation

$$\sigma \equiv \frac{z + h(x,y)}{d(x,y,t)}, \tag{6}$$

where $d(x,y,t) = \eta(x,y,t) + h(x,y)$ is the water column at any given time.

The spatial derivatives of the coordinate σ can be written as

$$\begin{aligned}\nabla\sigma &= \frac{1-\sigma}{d}\nabla h - \frac{\sigma}{d}\nabla\eta, \qquad \partial_z\sigma = \frac{1}{d},\\ \nabla^2\sigma &= \frac{1-\sigma}{d}\left(\nabla^2 h - \frac{\nabla h\cdot\nabla h}{d}\right) - \frac{\sigma}{d}\left(\nabla^2\eta - \frac{\nabla\eta\cdot\nabla\eta}{d}\right)\\ &\quad - \frac{1-2\sigma}{d^2}\nabla h\cdot\nabla\eta - \frac{\nabla\sigma}{d}\cdot(\nabla h + \nabla\eta).\end{aligned} \tag{7}$$

When the water waves evolve the physical domain changes, and this can be accounted for by relating the temporal derivative of the variables in the physical domain to those of the temporal derivatives in the σ-transformed domain via this relation, e.g.

$$\left.\frac{\partial\mathbf{u}}{\partial t}\right|_{(x,y,z)} = \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{(x,y,\sigma)} + \frac{\partial\sigma}{\partial t}\frac{\partial\mathbf{u}}{\partial\sigma}, \qquad \frac{\partial\sigma}{\partial t} = -\frac{\sigma}{d}\frac{\partial\eta}{\partial t}. \tag{8}$$

Hence, it is possible to derive the σ-transformed momentum equations

$$\rho\frac{\partial\mathbf{u}}{\partial t} = -\nabla_\sigma p + \mu\nabla_\sigma^2\mathbf{u} + \mathbf{F} - \rho(\mathbf{u}\cdot\nabla_\sigma)\mathbf{u} - \rho\frac{\partial\sigma}{\partial t}\frac{\partial\mathbf{u}}{\partial\sigma}, \quad t>0, \quad (x,y,\sigma)\in\Omega_\sigma. \tag{9}$$

Here we have introduced the σ-transformed gradient operator
$\nabla_\sigma = (\partial/\partial x + \partial\sigma/\partial x\,\partial/\partial\sigma, \partial/\partial y + \partial\sigma/\partial y\,\partial/\partial\sigma, \partial\sigma/\partial z\,\partial/\partial\sigma)$.

A Poisson problem can be defined for the pressure, and boundary conditions are required. Performing a projection of (2) in the (outward pointing) normal direction at a domain boundary $\Gamma\in\partial\Omega$, it is possible to derive a Neumann condition for the pressure

$$\mathbf{n}\cdot\nabla_\sigma p = \mu\mathbf{n}\cdot\nabla_\sigma^2\mathbf{u} + \mathbf{n}\cdot\mathbf{F} - \rho\mathbf{n}\cdot(\mathbf{u}\cdot\nabla_\sigma)\mathbf{u}, \quad \mathbf{x}\in\Gamma. \tag{10}$$

Remark, this condition can be rewritten into an inhomogeneous domain boundary condition for the dynamic pressure using (5). At the free surface, (5) defines the Dirichlet condition $p_D = 0$ Pa.

Furthermore, in our experiments, we assume solid walls (W) at all (fixed) domain boundaries $\Gamma^W\in\partial\Omega$, stated as

$$\mathbf{n}\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Gamma^W. \tag{11}$$

Numerical discretization

The high-order numerical scheme based on a finite difference method (FDM) and a method of lines discretization of the problem stated as a general initial value problem $d\mathbf{q}/dt = \mathbf{f}(\mathbf{q}, t)$ with initial condition $\mathbf{q}(t_0) = \mathbf{q}_0$. The numerical scheme implemented is inspired by the robust stencil-based numerical schemes for wave-structure interactions for high-order fully nonlinear and dispersive Bousinesq-type equations Engsig-Karup (2006) (cf. chap. 5, sec. 5.5.1), high-order finite difference stencils for fully nonlinear potential flow equations Bingham & Zhang (2007), and Engsig-Karup et al. (2009) where ghost nodes was introduced to discretise the boundary conditions while also satisfying the governing equations at all gridpoints.

A general explicit k-stage Runge-Kutta method is stated as

$$\mathbf{q}^{(k)} = \sum_{l=1}^{k} \alpha_{kl}\mathbf{q}^{l-1} + \beta_{kl}\Delta t\mathbf{f}(\mathbf{q}^{l-1}), \quad k = 1, 2, ..., s, \tag{12}$$

where $\mathbf{q}^{(0)} = \mathbf{q}^n$ is the current state at time $t_n = \Delta tn$, $\mathbf{q}^{(s)} = \mathbf{q}^{n+1}$ is the new state, and $\mathbf{f}$ is the right hand side vector derived from (3) and (9). The state vector is $\mathbf{q} = (\eta, \mathbf{u})$. By taking the divergence of the discrete (9) and require that it vanishes at the new state, a discrete mixed-stage Poisson problem for the dynamic pressure can be stated as

$$\nabla_\sigma^{(k)} \cdot \nabla_\sigma^{(k-1)} p_D^{(k-1)} = \frac{\rho}{\beta_{kk}\Delta t}\nabla_\sigma^{(k)} \cdot \hat{\mathbf{u}}^{(k)}, \tag{13a}$$

$$p_D = 0, \tag{13b}$$

$$\mathbf{n} \cdot \nabla_\sigma^{k-1} p_D^{k-1} = \frac{\rho}{\beta_{kk}\Delta t}\mathbf{n} \cdot \hat{\mathbf{u}}^{(k)}, \tag{13c}$$

$$\hat{\mathbf{u}}^{(k)} = \sum_{l=1}^{k-1} \alpha_{kl}\mathbf{u}^{(l-1)} + \beta_{kl}\Delta t\mathbf{f}(\mathbf{u}^{(l-1)}) + \alpha_{kk}\mathbf{u}^{(k-1)} + \beta_{kk}\Delta t\hat{\mathbf{f}}(\mathbf{u}^{(k-1)}). \tag{13d}$$

Here $\hat{\mathbf{f}}(\cdot)$ is the right hand side of (9) without the dynamic pressure term.

In the work of Li & Fleming (2001) an explicit projection method was used for the temporal time integration. We use the high-order scheme presented by Bitsch et al. (2023). A low-storage explicit 4th order 5-stage Runge-Kutta (LSERK45) method is used for temporal integration of the discrete governing equations (3) and (9) while enforcing at every stage the geometric conservation law (4) through solving a discrete pressure-velocity coupling similar to Chorin (1968) to determine the dynamic pressure that ensure satisfying the mass continuity condition every Runge-Kutta stage.

Numerical experiments

We present preliminary results of numerical experiments. In Figure 1 we demonstrate the ability to solve the inviscid Poisson problem for the dynamic pressure accurately for the steepest stream function waves (assume $\mu = 0$ mPA·s for inviscid fluid flow) in a periodic domain setup in 2D, i.e. with the horizontal boundaries assumed periodic. In Figure 2 we present numerical results of a full time-domain Navier-Stokes simulation obtained from the Beji & Battjes (1994) experiment for harmonic generation over a submerged bar for the case with an input wave of period $T = 2.02$ s.

In ongoing work, we aim to present numerical 3D experiments for wave-bottom and wave-structure interaction. Also, we will design new efficient multigrid schemes for large-scale simulations and accelerating the performance of the Navier-Stokes FDM solver.

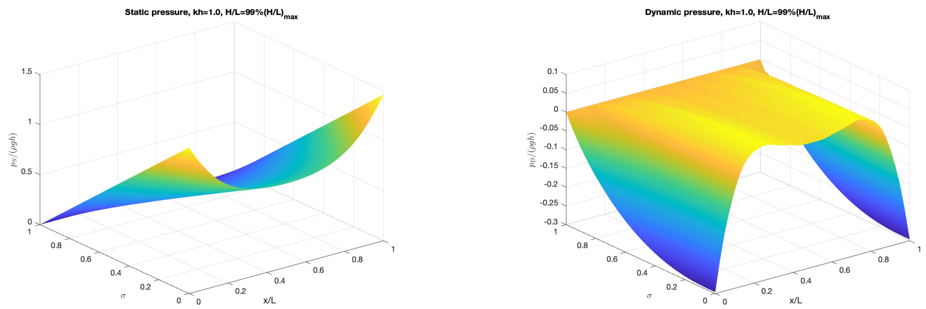

Figure 1: Computed normalized pressure distributions in the fluid volume below the free surface for a steep nonlinear streamfunction wave corresponding, $kh = 1$, $H/L = 99\%(H/L)_{max}$=0.0994. Left: static pressure. Right: dynamic pressure.

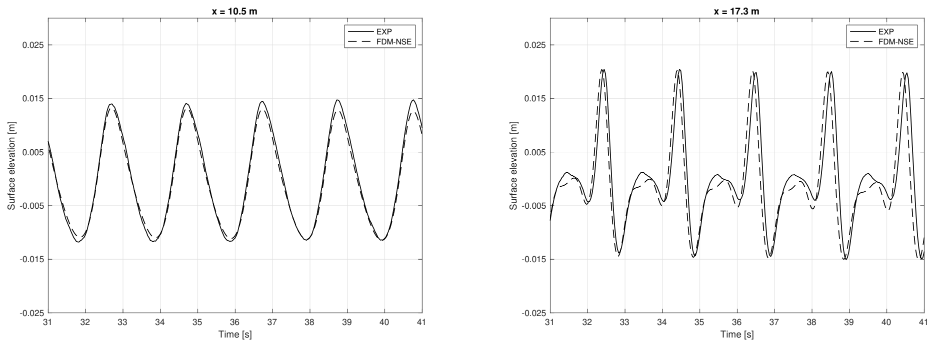

Figure 2: Simulation of nonlinear and dispersive wave propagation across a bar using a free surface Navier-Stokes FDM model. Left: wave elevation at gauge 10.5 m before the bar. Right: wave elevation at gauge 17.3 m after the bar.

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