# Free surface pressure and profile measurements from seabed pressure gauges

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## Highlights

Equations relating the pressure at a horizontal seabed, the free-surface profile and the surface-pressure are derived for two-dimensional irrotational steady water waves with arbitrary pressure at the free surface. Special cases include gravity, capillary, flexural and wind waves.

## 1 Introduction

The recovery of pure gravity (i.e., with constant surface-pressure) irrotational steady waves from bottom pressure gauges as a long been proposed. These methods either solve the problem exactly or under various assumptions; see [1, 2, 3, 4] and the references therein for details. Recently, it was shown that an exact recovery is also possible in presence of constant vorticity [5]. However, to the authors knowledge, the recovery of capillary, flexural and wind waves (among many other situations of physical interest) has never been attempted. These phenomena involve different non-constant surface-pressures that can be very complicated (especially for capillary and flexural waves). Here, we describe a general recovery method valid for any surface-pressure, allowing to recover both the surface-profile and the surface-pressure.

## 2 Equations of motion

In the frame of reference moving with a traveling wave of permanent shape, the flow beneath the wave is a steady two-dimensional irrotational motion of an inviscid fluid. Let $(x, y)$ be a Cartesian coordinate system moving with the wave, $x$ being the horizontal coordinate and $y$ the upward vertical coordinate and let $(u(x, y),\, v(x, y))$ be the velocity field in this moving frame. We denote by $y = -d$, $y = \eta(x)$ and $y = 0$ the equations of the bottom, of the free surface and of the mean water level, respectively. The latter equation expresses that $\langle\eta\rangle = 0$ for a smooth $(2\pi/k)$-periodic wave profile $\eta$, where $\langle\cdot\rangle$ is the Eulerian average operator over one period, i.e.

$$\langle\, \eta\, \rangle \;\stackrel{\text{def}}{=}\; \frac{k}{2\,\pi} \int_{-\pi/k}^{\pi/k} \eta(x)\; \mathrm{d}x \;=\; 0. \tag{1}$$

For solitary and other aperiodic waves, the same averaging operator applies taking the limit $k \to 0^+$. The flow is governed by the balance between the restoring gravity force, the inertia of the system and a surface pressure. With constant density $\rho > 0$ and acceleration due to gravity $g > 0$, the kinematic and dynamic equations are, for $x \in \mathbb{R}$ and $y \in [-d; \eta(x)]$,

$$u_x + v_y \;=\; 0, \qquad v_x - u_y \;=\; 0, \qquad u^2 + v^2 + 2\,g\,y \;=\; -2\,p, \tag{2a, b, c}$$

where $p(x, y)$ denotes the physical pressure divided by the density and $B$ is a Bernoulli constant.

The flat bottom and the wavy free surface being impermeable, we have $v_\mathrm{b} = 0$ and $v_\mathrm{s} = u_\mathrm{s}\eta_x$ with $\eta_x \stackrel{\text{def}}{=} \mathrm{d}\eta/\mathrm{d}x$ and where subscripts 'b' and 's' denote, respectively, restrictions at the bottom and at the free surface, e.g. $u_\mathrm{b}(x) = u(x, -d)$, $v_\mathrm{s}(x) = v(x, \eta(x))$. The pressure at the free surface $p_{\rm s}$ can be zero or a varying if, for instance, it models a prescribed surface (wind effect) or capillary and flexural effects such that

$$p_{\rm s} \,=\, -\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\tau\,\eta_x}{(1+\eta_x^2)^{1/2}} - \frac{D\,\eta_{xxx}}{(1+\eta_x^2)^{5/2}} + \frac{5\,D\,\eta_x\,\eta_{xx}^2}{2\,(1+\eta_x^2)^{7/2}}\right\}, \tag{3}$$

$\tau$ being a surface tension coefficient and $D$ a rigidity parameter (both divided by the fluid density). We take $\langle p_{\rm s}\rangle = 0$ without loss of generality, since $\langle p_{\rm s}\rangle$ can be absorbed into the definition of the atmospheric pressure. Thus, from the definition (1) of the mean level, one gets [2, 5]

$$B \,=\, \left\langle\, u_{\rm s}^2 + v_{\rm s}^2 \,\right\rangle \,=\, \left\langle\, u_{\rm b}^2 \,\right\rangle, \tag{4}$$

yielding the, here important, relation $\langle\, p_{\rm b}\,\rangle = g\,d$. Finally, equations (2*a*-*b*) imply that the complex velocity $w \stackrel{\text{def}}{=} u - \mathrm{i}v$ is a holomorphic function of $z \stackrel{\text{def}}{=} x + \mathrm{i}y$.

## 3 Equations for the free-surface and surface-pressure recoveries

The function $(u - \mathrm{i}v)^2$ being holomorphic, its real and imaginary parts satisfy the Cauchy–Riemann relations

$$\partial_y\left[u^2 - v^2\right] - \partial_x\left[2\,u\,v\right] \,=\, 0, \qquad \partial_x\left[u^2 - v^2\right] + \partial_y\left[2\,u\,v\right] \,=\, 0. \tag{5$a,b$}$$

Integrating over the water column and using the boundary conditions, these relations yield

$$p_{\rm b} - p_{\rm s} - g\,h \,=\, \frac{\mathrm{d}}{\mathrm{d}x}\int_{-d}^{\eta} uv\,\mathrm{d}y, \qquad (p_{\rm s} + g\,\eta)\,\frac{\mathrm{d}\,\eta}{\mathrm{d}x} \,=\, \frac{\mathrm{d}}{\mathrm{d}x}\int_{-d}^{\eta}\frac{u^2 - v^2 + B}{2}\,\mathrm{d}y. \tag{6$a,b$}$$

Taylor expansions around $y = -d$ can be written

$$u^2 - v^2 = \cos[(y+d)\partial_x]\,u_{\rm b}^2 = -2\cos[(y+d)\partial_x]\,(p_{\rm b} - gd), \tag{7}$$

$$2uv = -\sin[(y+d)\partial_x]\,u_{\rm b}^2 = 2\sin[(y+d)\partial_x]\,(p_{\rm b} - gd). \tag{8}$$

Hence, with $h \stackrel{\text{def}}{=} d + \eta$, we have

$$\int_{-d}^{\eta} uv\mathrm{d}y = \left[1 - \cos(h\partial_x)\right]\partial_x^{-1}\left(p_{\rm b} - gd\right), \tag{9a}$$

$$\int_{-d}^{\eta}\frac{u^2 - v^2 + B}{2}\mathrm{d}y = -\sin(h\partial_x)\,\partial_x^{-1}(p_{\rm b} - gd), \tag{9b}$$

so equations (6) yield

$$p_{\rm s} + g\eta = \partial_x\cos(h\partial_x)\,\partial_x^{-1}\left(p_{\rm b} - gd\right) = \left[\cos(h\partial_x) - \eta_x\sin(h\partial_x)\right](p_{\rm b} - gd), \tag{10}$$

$$(B - p_{\rm s} - g\eta)\eta_x = \partial_x\sin(h\partial_x)\,\partial_x^{-1}\left(p_{\rm b} - gd\right) = \left[\sin(h\partial_x) + \eta_x\cos(h\partial_x)\right](p_{\rm b} - gd). \tag{11}$$

After one integration, equation (11) becomes

$$B\eta - \tfrac{1}{2}g\eta^2 - \partial_x^{-1}\left(p_{\rm s}\eta_x\right) = \sin(h\partial_x)\,\partial_x^{-1}(p_{\rm b} - gd). \tag{12}$$

With the special surface pressure (3) we have

$$\partial_x^{-1}\left(p_{\rm s}\eta_x\right) = \frac{\tau}{(1+\eta_x^2)^{1/2}} - \tau + \frac{D\eta_x\eta_{xxx} - 3D\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} + \frac{5D\eta_{xx}^2}{2\,(1+\eta_x^2)^{7/2}} + \text{constant}, \tag{13}$$

where the integration constant must be determined by the mean level condition (1).

When $p_\mathrm{s}=0$ (pure gravity waves), $\eta$ can be obtained from $p_\mathrm{b}$ solving the ordinary differential equation (11) [2] or, more easily, solving the algebraic equation (12) [1]. When $p_\mathrm{s}\neq 0$ is a function of $x$ and/or $\eta$, such as (3), in general (12) is a complicated highly-nonlinear high-order integro-differential equation for $\eta$ due to the term $\partial_x^{-1} p_\mathrm{s}\eta_x$ (see relation (13) for an example of practical interest). This is not a problem for recovering the free surface $\eta$ from the bottom pressure $p_\mathrm{b}$ because the surface pressure $p_\mathrm{s}$ can be eliminated between (10) and (11), yielding

$$B\,\eta_x \;=\; \left\{\left(1-\eta_x^2\right)\sin[h\partial_x] + 2\,\eta_x\cos[h\partial_x]\right\}(p_\mathrm{b}-gd), \tag{14}$$

or in complex form — introducing $\widetilde{\mathfrak{P}}(z)\stackrel{\text{def}}{=}p_\mathrm{b}(z+\mathrm{i}d)-gd$ —

$$B\,\eta_x \;=\; \left(1-\eta_x^2\right)\mathrm{Im}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\} + 2\,\eta_x\,\mathrm{Re}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\}, \tag{15}$$

that is a (nonlinear) first-order ordinary differential equation for $\eta$. Equation (15) being algebraically quadratic for $\eta_x$, it can be solved explicitly for $\eta_x$, thus one gets

$$\mathrm{Re}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\} - \eta_x\,\mathrm{Im}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\} \;=\; \tfrac{1}{2}B \;\pm\; \tfrac{1}{2}\left|\,B-2\,\widetilde{\mathfrak{P}}_\mathrm{s}\,\right|. \tag{16}$$

Since the free surface is flat if the bottom pressure is constant (and since $B>0$), the minus sign must be chosen. Moreover, the condition (4) rewritten in terms of $\widetilde{\mathfrak{P}}$ yielding $B=\left\langle\,\left|B-2\,\widetilde{\mathfrak{P}}_\mathrm{s}\right|\,\right\rangle$, the average of the right-hand side of (16) is zero, so is the left-hand side.

Equation (16) is *a priori* not suitable if $\eta$ is (nearly) not differentiable (limiting waves). It is thus more efficient to solve its antiderivative

$$2\;\mathrm{Re}\left\{\widetilde{\mathfrak{Q}}_\mathrm{s}\right\} - K \;=\; \partial_x^{-1}\left[\,B-\left|\,B-2\,\widetilde{\mathfrak{P}}_\mathrm{s}\,\right|\,\right], \tag{17}$$

where $K$ is an integration constant and where $\widetilde{\mathfrak{Q}}(z)\stackrel{\text{def}}{=}q_\mathrm{b}(z+\mathrm{i}d)$ with $q_\mathrm{b}(x)\stackrel{\text{def}}{=}\partial_x^{-1}(p_\mathrm{b}(x)-gd)$ choosing $\langle q_\mathrm{b}\rangle\stackrel{\text{def}}{=}0$, so $\partial_x\,\mathrm{Re}\left\{\widetilde{\mathfrak{Q}}_\mathrm{s}\right\}=\mathrm{Re}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\}-\eta_x\,\mathrm{Im}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\}$ and $\left\langle(1+\mathrm{i}\eta_x)\widetilde{\mathfrak{Q}}_\mathrm{s}\right\rangle=0$. The right-hand side of (17) being the antideirative of a zero-average quantity, we conveniently choose $\left\langle\partial_x^{-1}\left[B-\left|B-2\widetilde{\mathfrak{P}}_\mathrm{s}\right|\right]\right\rangle\stackrel{\text{def}}{=}0$, hence $K=2\left\langle\mathrm{Re}\left\{\widetilde{\mathfrak{Q}}_\mathrm{s}\right\}\right\rangle$. Thus, a numerical resolution of (17) does not require the computation of $\eta_x$, that is an interesting feature for steep waves.

The free-surface $\eta$ being obtained after the resolution of (16) or (17), the surface-pressure $p_\mathrm{s}$ is obtained explicitly at once from (10)

$$p_\mathrm{s} \;=\; \partial_x\,\mathrm{Re}\left\{\widetilde{\mathfrak{Q}}_\mathrm{s}\right\} - g\eta \;=\; \mathrm{Re}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\} - \eta_x\,\mathrm{Im}\left\{\widetilde{\mathfrak{P}}_\mathrm{s}\right\} - g\eta. \tag{18}$$

Thus, as $\eta$, $p_\mathrm{s}$ is known modulo the Bernoulli constant $B$ that is the only quantity left to be determined. In order to fully recover both the free-surface and the surface-pressure, knowing only the bottom pressure is not sufficient so at least one extra information is needed. We consider here two possibilities of practical interest.

A first possibility is when we have access to one independent extra measurement, for instance the mean velocity at the bottom (or elsewhere), the mean pressure somewhere at a point above the seabed, the phase speed, the wave height, etc. In that case, the Bernoulli constant $B$ is chosen such that the recovered wave matches this measurement.

If no extra measurements are available, the free-surface can nevertheless be fully recovered with the knowledge (or reasonable guess) of the physical nature of the surface-pressure, for instance given by (3). The missing parameter can then be obtained minimising the error $\left\langle\,|p_{\mathrm{s}r}-p_{\mathrm{s}t}|^2\,\right\rangle$ between the recovered surface-pressure $p_{\mathrm{s}r}$ obtained from (18) and the theoretical surface-pressure $p_{\mathrm{s}t}$ given, say, by (3).

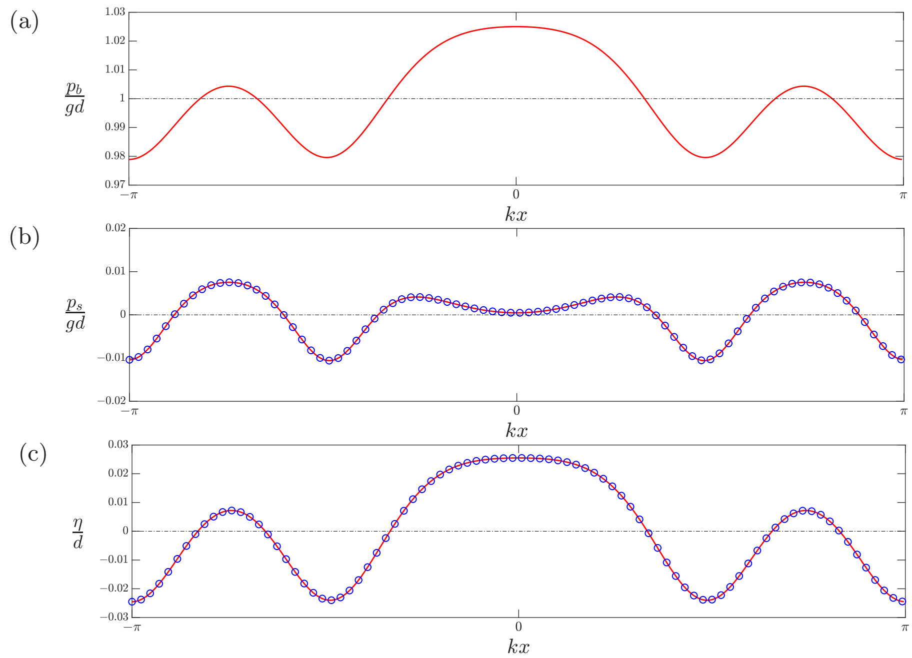


Figure 1: Recovery of a capillary-gravity wave with period $L/d = 6\pi$, Froude number square $B/gd = 1.01568$ and Bond number $\tau/gd^2 = 1/3$. (a): Bottom pressure treated as a "measurement" for the recovery procedure. (b,c): Respectively, recovered surface pressure and profile (blue circles) versus the exact solution (red line).

# 4 Summary

We described a general method for recovery the surface-profile and the surface-pressure from bottom-pressure measurements. An example of surface-profile and surface-pressure recoveries is given in Figure 1; interested readers can find more details in [6]. The approach can be generalised to flows with constant vorticity along the line of [5]. The approach can also be further generalised to handle overturning waves, as described in [7].

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