A force model for non-slender bodies in fully nonlinear waves

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Fully nonlinear wave kinematics can today be computed at low cost. While the forces on slender bodies can be found through simple post processing, there is presently no similar simple way to obtain the forces on non-slender bodies. We here devise a closed form force model for non-slender bodies, based on first-order radiation-diffraction output only. The method retains full nonlinearity in the incident waves below the still water level, and is consistent with the Pinkster approximation.

1 FIRST-ORDER LOADS

We first consider linear 3D incident waves with free surface elevation and velocity potential given by

$$\eta_I = \sum_{j=-N}^{N} A_j \mathrm{e}^{\mathrm{i}(\omega_j t - \mathbf{k}_j \cdot \mathbf{x})} \quad , \quad \phi_I = \sum_{j=-N}^{N} B_j \frac{\cosh k_j(z+h)}{\cosh k_j h} \mathrm{e}^{\mathrm{i}(\omega_j t - \mathbf{k}_j \cdot \mathbf{x})} \tag{1}$$

$$\mathbf{k}_j = k_j \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad , \quad \omega_j^2 = g k_j \tanh k_j h \tag{2}$$

where $A_{-j} = A_j^*,\ B_{-j} = B_j^*,\ k_{-j} = -k_j$ and $\omega_{-j} = -\omega_j$. The potential flow satisfies the Bernoulli equation, which at the free surface provides the linear relations between A_j and B_j

$$\phi_t + \frac{p}{\rho} + gz + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi = C(t) \tag{3}$$

$$B_j = \frac{\mathrm{i}g}{\omega_j} A_j \qquad A_j = -\frac{\mathrm{i}\omega_j}{g} B_j \tag{4}$$

Due to the presence of the structure, the total velocity potential consists of the incident potential plus the scattered potential

$$\phi = \phi_I + \phi_S \tag{5}$$

Following standard radiation-diffraction theory, the first-order force is obtained through integration of the linear pressure field over the body surface up to $z = 0$, which we denote S_0

$$\mathbf{F}_1 = \int_{S_0} p \,\mathrm{d}\vec{\mathbf{n}} = -\rho \int_{S_0} \phi_t \,\mathrm{d}\vec{\mathbf{n}} = -\rho \frac{\partial}{\partial t} \sum_{j=-N}^{N} \int_{S_0} \left[B_j \frac{\cosh k_j(z+h)}{\cosh k_j h} \mathrm{e}^{\mathrm{i}(\omega_j t - \mathbf{k}_j \cdot \mathbf{x})} + \phi_S \right] \mathrm{d}\vec{\mathbf{n}} \tag{6}$$

$$= \sum_{j=-N}^{N} \mathbf{X}(\omega_j, \theta_j) A_j \mathrm{e}^{\mathrm{i}\omega_j t} = \sum_{j=-N}^{N} \mathbf{X}(\mathbf{k}_j) A_j \mathrm{e}^{\mathrm{i}\omega_j t} \tag{7}$$

This well known result defines the transfer function for the excitation force $\mathbf{X}(\omega, \theta)$ which can also be expressed as a function of the wave number vector $\mathbf{k}$. Through (4), we can further recast (7) into

$$\mathbf{F}_1 = \frac{\partial}{\partial t} \sum_{j=-N}^{N} \frac{-1}{g} \mathbf{X}(\mathbf{k}_j) B_j \mathrm{e}^{\mathrm{i}\omega_j t} \tag{8}$$

2 FULLY NONLINEAR WAVES AND THE POTENTIAL FORCE ON S_0

For a fully nonlinear wave field at constant depth, the free surface elevation and velocity potential at the still water level, can for any instant of time be written as a Fourier series in the wave number vector space $\mathbf{k}$

$$\eta_I(\mathbf{x}, t^*) = \sum_{\mathbf{k}} \hat{A}(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \quad , \quad \Phi_I(\mathbf{x}, t^*) = \sum_{\mathbf{k}} \hat{B}(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{9}$$

The full spatial variation of the incident velocity potential is then

$$\phi_I(\mathbf{x}, z, t^*) = \sum_{\mathbf{k}} \hat{B}(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \frac{\cosh|\mathbf{k}|(z+h)}{\cosh|\mathbf{k}|h} \tag{10}$$

and contains both free and bound waves. Since the spatial structure of each component in ϕ_I is identical to the one considered in (1), the force associated with the potential is given by

$$\mathbf{F}_P = \frac{\partial}{\partial t} \sum_{\mathbf{k}} \frac{-1}{g} \mathbf{X}(\mathbf{k}) \hat{B}(\mathbf{k}) \tag{11}$$

As a difference from (8) to (11), we note that $\hat{B}(\mathbf{k})$ in (11) is a function of time, while for a purely linear wave field, B_j is constant and the time variation is explicitly known. For use with fully nonlinear kinematics, the values of $\hat{B}(\mathbf{k})$ can be obtained by FFT operation on the spatial snapshots of Φ_I at $z = 0$, such that the sum in (11) can be computed in every time step. Next, the resulting time series, which is the force impulse, can be differentiated to yield the force history. The subscript P denotes that we have here calculated the potential force. The result is consistent with Pinkster (1980) which neglects the nonlinear interaction between the incident and scattered wave field and the scattered wave field with itself.

3 QUADRATIC PRESSURE FORCE ON S_0

Beyond the force from ϕ_t, there is also a force from the squared velocity in the Bernoulli equation

$$\mathbf{F}_{QP} = -\tfrac{1}{2}\rho \int_{S_0} \nabla(\phi_I + \phi_S) \cdot \nabla(\phi_I + \phi_S) \, \mathrm{d}\vec{\mathbf{n}} \tag{12}$$

Given that $\nabla(\phi_I + \phi_S)$ at any point can be expressed from $\hat{B}(\mathbf{k})$ through a transfer function from the linear radiation-diffraction output, this force can be calculated by numerical integration over S_0 for any pair of $\mathbf{k}$'s and thus $\mathbf{F}_{QP}$ be expressed through a quadratic transfer function (QTF) in the $\hat{B}$'s. Hereby (12) can be computed at every time step based on the Fourier decomposition (9). We note that such QTF calculations can be speeded up by the method of Bredmose & Pegalajar-Jurado (2021).

4 LOADS BEYOND S_0

The load contribution between the instantaneous free surface and $z = 0$ is given by

$$F_{S-S_0} = -\rho \int_{WL} \int_{z=0}^{\eta_I+\eta_S} \left(\phi_t + gz + \tfrac{1}{2}\nabla\phi \cdot \nabla\phi\right) \left(1 - n_z^2\right)^{-1/2} \vec{\mathbf{n}} \, \mathrm{d}z \, \mathrm{d}l. \tag{13}$$

This force is less easy to calculate in closed form from the linear radiation diffraction solution only, which is limited to S_0. To keep a closed form format, we therefore need to apply some approximations. We pursue here a Taylor expansion to second order from $z = 0$, as also done for standard QTF's of radiation-diffraction theory (Lee 1995), but driven with the fully nonlinear wave input. The second-order approximation allows us to neglect the quadratic pressure term to obtain

$$\mathbf{F}_{WL} = -\rho \int_{WL} (\eta_I + \eta_S) \left[(\phi_{I,t} + \phi_{S,t}) + \tfrac{1}{2} g(\eta_I + \eta_S)\right] \left(1 - n_z^2\right)^{-1/2} \vec{\mathbf{n}} \, \mathrm{d}l \tag{14}$$

which similarly to (12) can be expressed through QTFs in the $\hat{A}$'s and $\hat{B}$'s of (9). Alternative methods to the QTF approach here are explicit integration of the Froude Krylov force of the incident wave in each time step with an approximation for the added mass coefficient, or use of slender body theory. We note that extension of (14) to third order is doable with only a few extra terms.

5 VALIDATION AGAINST SECOND-ORDER DIFFRACTION THEORY

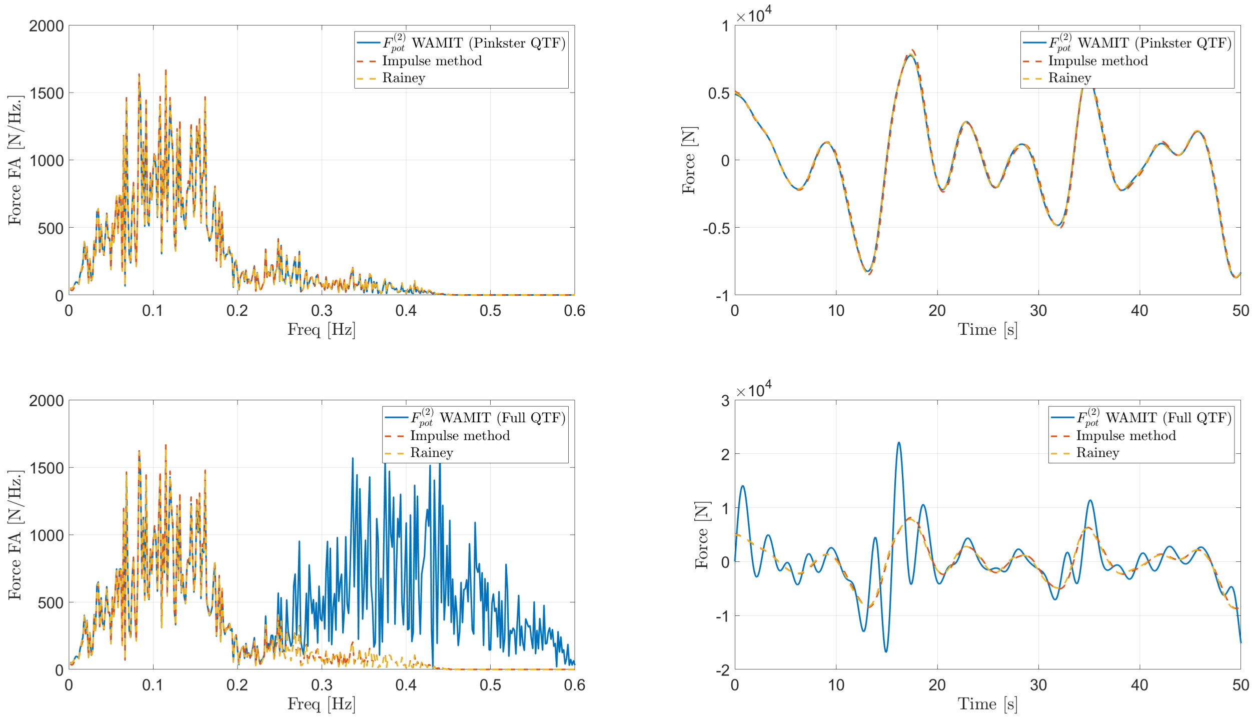

Figure 1: Top row: Comparison of power spectra (left) and time series excerpt (right) for the second-order potential force against radiation-diffraction QTF with exclusion of the free surface integral. Lower row: Comparison with full QTF.

As a first validation, we pursue a comparison to a well defined reference result. We thus validate the model against QTF calculations of radiation-diffraction theory, obtained by Wamit. We consider a vertical bottom mounted circular cylinder of diameter $D = 3$ m, on a depth of $h = 33$ m in an irregular JONSWAP sea state of $H_{m0} = 3.5$ m, $T_p = 7.5$ s and $\gamma = 3.3$. Since the reference solution is of second-order accuracy, we can apply closed form Stokes second-order wave theory as input to the new force model. In the notation of (1), the second-order potential is given by

$$\phi_{2,I} = \mathrm{i} \sum_{m=-N}^{N} \sum_{n=-N}^{N} B_m B_n \mathbb{T}_\Phi \mathrm{e}^{\mathrm{i}((\omega_m+\omega_n)t-(\mathbf{k}_m+\mathbf{k}_n)\cdot\mathbf{x})} \frac{\cosh(|\mathbf{k}_m + \mathbf{k}_n|)(z+h)}{\cosh|\mathbf{k}_m + \mathbf{k}_n|h} \tag{15}$$

where $\mathbb{T}_\Phi$ is the second-order transfer function from the dimensional Fourier amplitudess of the potential at $z = 0$. We can thus immediately write up the second-order potential force in closed form as

$$\mathbf{F}_{2P} = = -\frac{\mathrm{i}}{g}\frac{\partial}{\partial t} \sum_{m=-N}^{N} \sum_{n=-N}^{N} B_m B_n \mathbb{T}_{\Phi\ m,n} \mathbf{X}(\mathbf{k}_m + \mathbf{k}_n) \mathrm{e}^{\mathrm{i}((\omega_m+\omega_n)t-(\mathbf{k}_m+\mathbf{k}_n)\cdot\mathbf{x})}. \tag{16}$$

In the top row of figure 1, the power spectrum and a time series excerpt of the second-order potential force from (16) and Wamit QTF analysis with omission of the free surface integral (Pinkster approximation) are compared. At the graphical level, a good match is seen, thus suggesting, at least for the tested geometry, that the new method provides the right potential force. The lower panel shows the comparison with the full QTF of Wamit, which includes the free surface integral. Below 0.2 Hz,

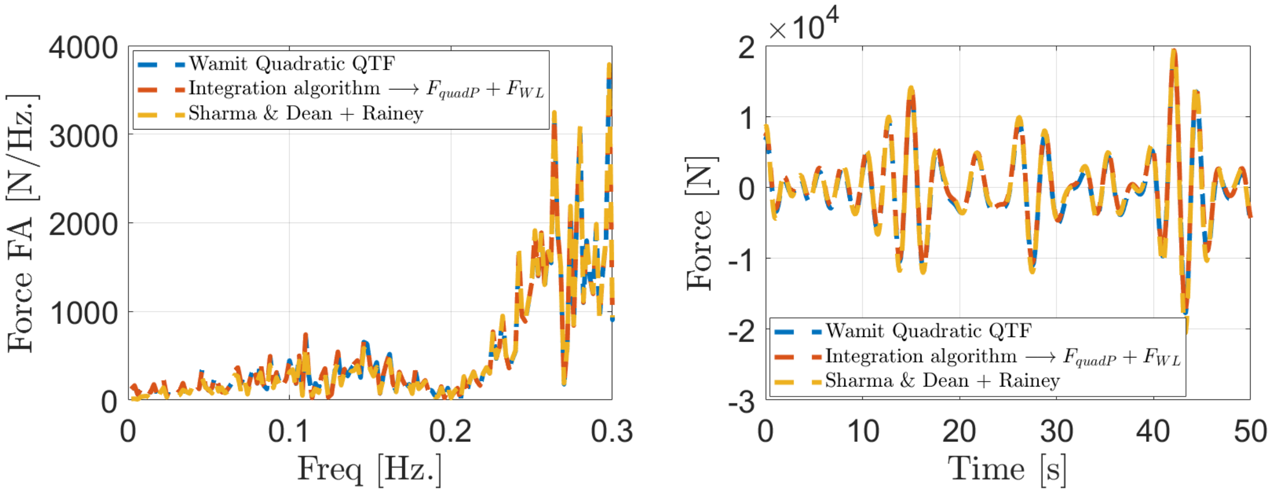

Figure 2: Comparison of the quadratic loads (quadratic pressure and water line) to QTF of radiation-diffraction theory.

the good match with the impulse method (16) is preserved, while the super harmonic range shows a severe under-prediction, due to the neglection of nonlinear interactions with the scattered waves.

We next compare the quadratic forces from the squared velocity term (12) and the water line integral (14) in figure 2. Also for the quadratic force a good match is seen both for the power spectrum and at the time series level. This provides a check, that the spatial integrations applied are consistent with the numerical reference solution.

For further comparison, we also include results of the Rainey (1995) force model applied to the inline force on the cylinder. Following the work in Bredmose & Pegalajar-Jurado (2021), the second-order terms from this model can be written as

$$F_{2I} = \rho\pi R^2 \left[\int_{-h}^{0} (C_m + 1)\left(u_{2,t} + u_1 u_{1,x} + w_1 u_{1,z}\right) + C_m w_{1,z} u_1 \mathrm{d}z + (C_m + 1)\eta_1 u_{1,t}|_{z=0} \right] \quad (17)$$

which contains the contributions from i) the second-order potential, ii) the convective acceleration, iii) the axial divergence force and iv) the water line integral. The associated results are included as yellow curves in figure 1-2 with the C_m values taken from MacCamy-Fuchs theory. The results are seen to match the new method and the reference solution very well. While this must be expected for the potential force due to the equivalence of the MacCamy-Fuchs and numerical radiation-diffraction solution, the good match for the quadratic terms is encouraging. Apparently, term (ii)-(iv) of the slender body model are able to represent the quadratic terms of the QTF similarly well as the new method, which may thus suggest a pragmatic alternative to (14). In summary, the proposed force model provides a closed-form approach to forces on non-slender bodies in fully nonlinear waves, presently consistent to second order. The extension to a moving body is part of our ongoing work.

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