Large time response of a floating viscoelastic plate in the vicinity of saddle point

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1 Introduction

In recent years, there has been notable interest in investigating flexural gravity waves caused by an oscillating and moving external pressure [6]. In the case of open water, it is known, see [1] and [4], that the linear wave theory provides unbounded water elevation for large times even for very small external pressure magnitude if the speed and the frequency of the external pressure are related in a certain way. For an external load moving along a thin floating elastic plate and oscillating such a relation is derived in this paper. It is also known that the singularity of the fluid response to a moving and oscillating pressure can be resolved with account for either viscous or nonlinear effects. We are concerned with the large-time response of a viscoelastic plate near the critical conditions in the presence of small damping. The critical conditions are related to the corresponding dispersion relation for the flexural-gravity waves, which has multiple propagating wave modes and double/triple roots [3]. This complex phenomena, where the group speed of some waves is very small, is known as wave blocking [3].

This study focuses on exploring the large-time asymptotic solution of plate deflection near the saddle point of the dispersion relation with account for viscous properties of the plate material.

2 Mathematical formulation

In this section, we delve into the mathematical modeling of an infinitely extended thin floating viscoelastic plate exposed to a time-harmonic external pressure moving at a constant speed along the plate. The viscoelastic properties of the plate are described by the Kelvin-Voigt model, see [5]. The problem is considered in a two dimensional Cartesian coordinate system moving together with the external pressure, where x-axis is in the direction of mean plate-covered surface and y-axis is acting vertically upward. It is assumed that the external load is moving at a constant speed U and its magnitude is oscillating at a frequency ω. The fluid is inviscid, incompressible and its flow is irrotational. Additionally, the floating plate is considered to be thin, homogeneous and isotropic. The water is of finite depth h, and the plate deflection is described by a function $\zeta(x,t)$. The plate deflection and the flow under plate are described within the linear theory of hydroelasticity. In the flow domain, the velocity potential $\Phi(x,y,t)$ satisfies the Laplace equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0, \qquad \text{for } -\infty<x<\infty,\ -h<y<0. \tag{1}$$

The rigid impermeable bottom bed condition is given by

$$\frac{\partial\Phi}{\partial y}=0, \qquad \text{on } y=-h. \tag{2}$$

The kinematic boundary condition on the plate/fluid interface yields

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial x}\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial y} \qquad \text{on } y = \zeta. \tag{3}$$

The equation of floating viscoelastic plate on $y = \zeta$ in the moving coordinates reads

$$EI\left(1 + \tau\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^4 \zeta}{\partial x^4} + \rho_p d\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \zeta = -\rho\left[\frac{\partial \Phi}{\partial t} + U\frac{\partial \Phi}{\partial x} + \right.$$
$$\left. \frac{1}{2}\left\{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right\} + g\zeta\right] + p(x)e^{\mathrm{i}\omega t} + c.c. \tag{4}$$

where the Bernoulli equation was used, the derivatives of velocity potential are evaluated at the current position of the plate surface, $y = \zeta(x,t)$, $p(x)$ is the magnitude and ω is the frequency of the external pressure, E is the Young's modulus, I is the moment of inertia, τ is the damping coefficient, ρ_p is the plate density and d is the plate thickness.

It is assumed that the external pressure distribution is switched on impulsively at $t = 0$ with water being undisturbed at $t \leq 0$, so that

$$\zeta = \zeta_t = 0, \quad \text{and} \quad \Phi = \Phi_t = 0. \tag{5}$$

3 Dispersion relation and wave blocking

Under the assumption of a plane wave solution of the form $\zeta(x,t) = \Re\{\zeta_0 e^{\mathrm{i}(kx+\omega t)}\}$ with k being the wavenumber associated with the frequency ω of external load, the linearised velocity potential is

$$\Phi(x,y,t) = a_1\frac{\cosh k(h+y)}{\cosh kh}e^{\mathrm{i}(kx+\omega t)}, \tag{6}$$

where $a_1 = (\mathrm{i}\Psi/\Omega)\zeta_0 - (Dk^4\tau)\zeta_0$ with $\Psi = (Dk^4 - m\Omega^2 + g)$ and $\Omega = \omega + kU$. In Eq. (6), the frequency ω and wavenumber k are related through the dispersion relation

$$(1 + \gamma k \tanh kh)\,\Omega^2 - \mathrm{i}(Dk^5 \tanh kh)\tau\Omega - (Dk^4 + 1)gk\tanh kh = 0, \tag{7}$$

with $D = EI/\rho g$ and $\gamma = \rho_p d/\rho g$. This is a complex dispersion relation due to the presence of damping constant τ. As a special case with $\tau = 0$ and $\gamma = 0$, Eq. (7) reduces to the plane flexural gravity wave dispersion relation with current [2].

Figure 1 demonstrates the dispersion curves for $\tau = 0$ as defined by Eq. (7) for different values of speed U. A noticeable trend emerges for wavenumbers $k < 0$, showing an increase in frequency regardless of uniform current U. However, an intriguing observation arises for the case of $U = 3.5$ m/s, where a portion of the dispersion curve with $k > 0$ becomes parallel to the horizontal axis. This implies that for $U > 3.5$ m/s the equation (7) considered in terms of k has several roots, two or three, for the same frequency ω. The dispersion curve for $U = 4.5$ m/s, for example, shows three roots, R_1, R_2 and R_3 for $\omega = 0.4$ 1/s, with a minimum and maximum between them. At the points, where the slope of dispersion curve for a certain value of the load speed U becomes zero, $d\omega/dk = 0$, the group velocity of the corresponding wave is zero and, therefore, the energy does not propagate from the moving load. The points on dispersion curves with $d\omega/dk = 0$ are called blocking points. For small positive τ, the complex roots of (7) can be close to each other in the way as the roots R_1, R_2 and R_3 shown in Fig. 1 merge for the speed U reducing down to 3.5 m/s.

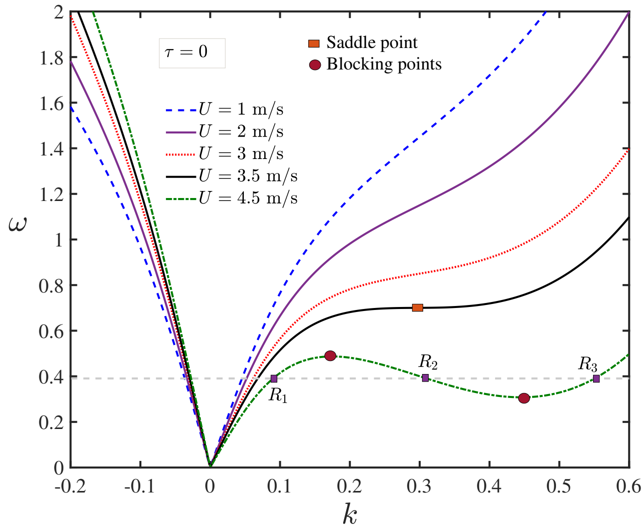

Figure 1: Dispersion curves for different values of speed U with $EI = 5 \times 10^8$ N m^{-2}, $d = 0.05$ m and $h = 10$ m.

4 Large time response of plate deflection near the saddle point

In order to investigate the linear large time response of a floating viscoelastic plate in a vicinity of saddle point where three positive roots of Eq. (7) merge, we assume a small plate deflection with scale η_{sc} and characteristic length scale $L_c = (D/\rho g)^{1/4}$. The hydrodynamic and structural parameters and variables are made non-dimensional using η_{sc} and L_c,

$$x = L_c\bar{x},\ y = L_c\bar{y},\ t = \bar{t}/\omega,\ \eta = \eta_{sc}\bar{\eta},\ \Phi = \omega\eta_{sc}L_c\bar{\Phi},\ p = p_{sc}\bar{p},$$
$$U = L_c\omega\bar{U},\ g = \omega^2\eta_{sc}\bar{g},\ \text{and}\ \tau = \omega\bar{\tau},$$

where $\eta_{sc} = p_{sc}/(\rho\omega^2 L_c)$. The dimensionless variables are denoted by the bar, which is dropped below. It is worth mentioning that the nonlinear terms in the dimensionless form of Eqs. (3) and (4) can be approximately neglected when $\varepsilon = \eta_{sc}/L_c \ll 1$. Next, by using the Fourier transformation, the transformed velocity potential of the linear problem is obtained as

$$\hat{\phi}(k,t) = \frac{\mathrm{i}\hat{p}(k)(1+kU)}{(f_+(k) - f_-(k))\cosh kh}\left[\frac{e^{\mathrm{i}t} - e^{-\mathrm{i}tf_-(k)}}{1+f_-(k)} - \frac{e^{\mathrm{i}t} - e^{-\mathrm{i}tf_+(k)}}{1+f_+(k)}\right], \tag{8}$$

with $f_\pm = \left\{kU - \mathrm{i}\tau\dfrac{Dk^5\tanh kh}{2(1+mk\tanh kh)}\right\} \pm \sqrt{\dfrac{(g+Dk^4)k\tanh kh}{1+mk\tanh kh} - \tau^2\left(\dfrac{Dk^5\tanh kh}{2(1+mk\tanh kh)}\right)^2}$, and $\hat{p}(k) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} p(x)e^{-\mathrm{i}kx}dx$. It can be easily checked that $1 + f_- = 0$ and $1 + f_+ = 0$ represent the positive and negative branches of the dispersion relation (7). According to the discussion made in Section 3, it is assumed that $1 + f_+(k_1) \approx 0$ and $1 + f_-(k_0) \approx 0$ with k_0 (saddle point) and $k_1(< 0)$ are the roots of multiplicity three and one respectively. The integral (8) can be decomposed as

$$\Phi(x,y,t) = I_+ + I_-, \tag{9}$$

where $I_{\pm} = \int_{-\infty}^{\infty} \frac{\mathrm{i}\hat{p}(k)(1+kU)}{2(f_+(k)-f_-(k))\cosh kh} \frac{e^{\mathrm{i}t} - e^{-\mathrm{i}tf_{\pm}(k)}}{1+f_{\pm}(k)} \frac{\cosh k(y+h)}{\cosh kh} e^{\mathrm{i}kx} dk$. Note that the integrals $I_{\pm}$ are not singular for $\tau > 0$. Next, in order to determine the large-time asymptotic of I_-, we consider $1 + f_-(k) = \mathcal{G}_1(k) - \mathrm{i}\tau\mathcal{G}_2(k)$ and the wavenumber k_0 being slightly perturbed, $k_0 = k_0^R + \mathrm{i}k_0^I$, for $\tau > 0$. After expanding $\mathcal{G}_1(k)$ about $k = k_0^R$, the integral I_- is rewritten as

$$I_- \approx \frac{1}{2} F_1(k_0^R) e^{\mathrm{i}(k_0^R x + t)} t^{2/3} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}\alpha x/t^{1/3}}}{\alpha^3 \mathcal{G}_1'''(k_0^R) - \mathrm{i}\tau t \mathcal{G}_2(k_0^R)} \left\{ 1 - e^{-\mathrm{i}\{\alpha^3 \mathcal{G}_1'''(k_0^R) - \mathrm{i}\tau t \mathcal{G}_2(k_0^R)\}} \right\} d\alpha,$$

where $\alpha = \xi t^{1/3}$ with $\xi = (k - k_0^R)/\sqrt[3]{6}$ and

$$F_1(k_0^R) = \frac{\mathrm{i}\hat{p}(k_0^R)(1 + k_0^R U)}{\{f_+(k_0^R) - f_-(k_0^R)\}\cosh k_0^R h} \frac{\cosh k_0^R (y+h)}{\cosh k_0^R h}.$$

Using contour integration, it can be shown that the integral on the right hand side tends to zero as $|x/t^{1/3}| \to \infty$, see [1] for open water case. Correspondingly, the asymptotic behavior of the plate deflection is obtained as

$$\zeta(x,t) \sim f_1(k_0^R) \frac{\mathrm{i}[D\{k_0^{R^4}(1 + \mathrm{i}\tau(1 + Uk_0^R))\} - m(1 + k_0^R U)^2 + g]}{1 + k_0^R U} e^{\mathrm{i}\theta} t^{2/3} I(x/t^{1/3}, \tau t),$$

with $\theta = k_0^R x + t$,

$$f_1(k_0^R) = \frac{2\mathrm{i}\hat{p}(k_0^R)(1 + k_0^R U)}{\{f_+(k_0^R) - f_-(k_0^R)\}\cosh k_0^R h},$$

and I be the integral on the right hand side of I_-. Therefore, the deflection of the floating viscoelastic plate depends on time t and grows like $t^{2/3}$ in the far-field (x/t fixed and $t \to \infty$) for $\tau t = O(1)$. For $t \to \infty$ and $\tau t \to 0$, the plate response is independent of plate damping for large times, see [1]. For $t \to \infty$ and $\tau t \to \infty$, no wave propagation occurs in the far-field.

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