High-order surface integration methods for an unstructured triangulation of the ship geometry

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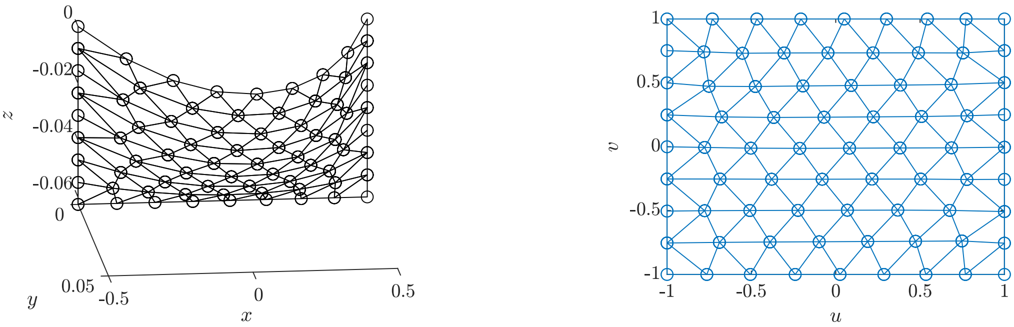
1 Introduction

Any high-order hydrodynamic solver requires a high-order surface integration scheme to compute the forces and moments on the hull. This work is motivated by our experience with the native surface integration scheme from the *Overture* library [3], upon which our seakeeping and added resistance solver *OceanWave3D-Seakeeping* [1] is built. Within this framework, the geometry is described by a collection of boundary-fitted, overlapping grids and the numerical solution is obtained using fourth-order accurate finite difference schemes. Nevertheless, surface integrals using the built-in library functions show relatively large errors unless the body geometry is very simple. In this abstract, we present two high-order methods for performing accurate surface integration on a scattered point set representing the body surface geometry. Although the primary motivation is to improve the integration methods within OceanWave3D-Seakeeping and Overture, the methods can be applied to any solver where the solution can be provided at a scattered collection of points on the body surface.

2 Surface integration on a scattered point set



(a) The physical grid (not to scale).

(b) The $u - v$ parameter space grid.

Figure 1: An example grid over a Wigley hull.

We assume that the body surface S is represented by the paramatrized physical coordinates $\mathbf{x} = [x(u,v), y(u,v), z(u,v)]$, where the $u-v$ parameter-space is the unit square R. The integral of the function $f(\mathbf{x})$ over the surface is given by

$$I = \int_S f(\mathbf{x})\, dS = \int_R f(u,v)\, |\mathbf{x}_u \times \mathbf{x}_v|\, dA, \tag{1}$$

where $\mathbf{x}_u$ indicates the derivative of the coordinate vector with respect to u (similarly for v), and dA is the element of area in the u,v parameter space. A visualization using the Wigley hull is shown in Fig. 1 with the mapping $u = x/L$, $v = -1 - z/D$ where L and D are the ship length and draft respectively. After mapping the discrete physical points to the $u - v$ plane, Delauney triangulation is used to generate a grid of triangles, as indicated in the figure. The first fundamental form [2] of the surface S can be written

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v, \\ H &= \sqrt{EG - F^2} = |\mathbf{x}_u \times \mathbf{x}_v|, \quad \mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{H}, \end{aligned} \tag{2}$$

where $\mathbf{n}$ is the unit normal vector to the surface. The integral I in Eq. (1) is now given by

$$I = \sum_{k=1}^{N_{el}} I_k, \qquad I_k = \int_{R_k} g(u, v) du\, dv, \tag{3}$$

where $g = f\, H$, N_{el} is the total number of triangles and R_k is the k^{th} triangle surface. Next, we map each triangle to the standard triangle with vertices at $(\xi, \eta) = [(0, 0), (1, 0), (0, 1)]$ using

$$\begin{aligned} u &= u_1 + \xi \Delta u_1 + \eta \Delta u_2 \\ v &= v_1 + \xi \Delta v_1 + \eta \Delta v_2, \\ & \Delta u_1 = u_2 - u_1, \quad \Delta u_2 = u_3 - u_1, \quad \Delta v_1 = v_2 - v_1, \quad \Delta v_2 = v_3 - v_1, \end{aligned} \tag{4}$$

where $[u_i, v_i]$, $i = 1, 2, 3$ are the coordinates of the three vertices on triangle k. The Jacobian of this transformation gives the triangle area, $A_k = \Delta u_1 \Delta v_2 - \Delta u_2 \Delta v_1$. Finally, we can write the integral over the k^{th} triangle in Eq. (3) as

$$I_k = A_k \int_0^1 \int_0^{1-\xi} g(u, v)\, d\eta\, d\xi. \tag{5}$$

2.1 Gaussian quadrature

Having mapped the triangle to the standard triangle, we can apply standard Gaussian quadrature schemes to approximate the integral in Eq. (5) by

$$I_k \approx A_k \sum_{j=1}^{N_g} w_j\, g(\xi_j, \eta_j), \tag{6}$$

where N_g is the number of Gauss points, w_j are the weights, and (ξ_j, η_j) are the Gauss point coordinates. Here we apply the 4th-order, 7-point scheme from [4]. To evaluate the function $g(u, v)$ at the Gauss point locations, we apply an order p Weighted Least Squares (WLS) scheme using $(p + 3)^2$ nearest neighbors and a Gaussian distance weighting. WLS is a generalization of the finite difference method which can be applied to a scattered point set, see for example [5] for a detailed description. The WLS scheme also provides an order p approximation for the first-derivatives required to evaluate the element of area and the normal vector in Eq. (2). Adopting a pseudo-code notation, we express the vector of function values at each of the N points on the grid as $\hat{g}(1 : N, 1)$, and the WLS interpolation vector for Gauss point j on element k is denoted $D_{jk}^{(0,0)}(1, 1 : N)$. Now the integral approximation is given by

$$\begin{aligned} I &= \left[\sum_{k=1}^{N_{el}} A_k \sum_{j=1}^{N_g} w_j\, D_{jk}^{(0,0)}(1, 1 : N) \right] \hat{g}(1 : N, 1) \\ &= \hat{I}_\xi(1, 1 : N) * \hat{g}(1 : N, 1) = \hat{I}(1, 1 : N) * \hat{f}(1 : N, 1), \end{aligned} \tag{7}$$

where the $*$ represents a discrete vector dot product and the accumulated sum in the brackets results in an *integration vector* which can be built once for a particular grid and subsequently used to efficiently calculate the integral of any function over the body surface. Here, $\hat{I} = \hat{I}_\xi .* \hat{H}^T$, where $\hat{H}(1 : N, 1)$ are the discrete values of H at each point, the superscript T indicates the vector transpose and $.*$ indicates an element-wise product.

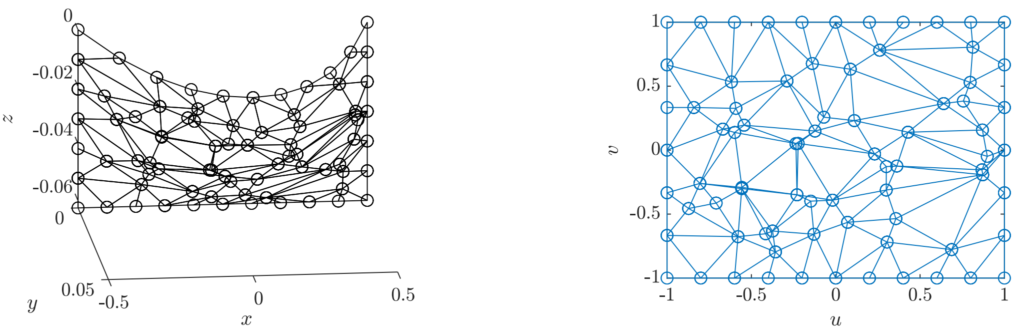
2.2 Taylor integration

An alternative to Gaussian quadrature is what we call *Taylor integration*. The idea here is to take the Taylor series expansion (truncated to order p) of $g(u,v)$, about the first vertex of the triangle (u_1,v_1), and insert it into Eq. (5), which gives

$$\begin{aligned} I_k &= A_k \sum_{m=0}^{p}\sum_{n=0}^{p} g_k^{(m,n)} \int_0^1 \int_0^{1-\xi} \frac{(\Delta u_1 \xi + \Delta u_2 \eta)^m (\Delta v_1 \xi + \Delta v_2 \eta)^n}{m!n!}\, d\eta\, d\xi \\ &= A_k \sum_{m=0}^{p}\sum_{n=0}^{p} \frac{g_k^{(m,n)}}{(m+n+2)(m+1)!n!} \Bigg[\Delta u_1^{m+1} \Delta v_1^{n+1} {}_2F_1 \left([1, m+n+2], m+2, \frac{\Delta u_1(\Delta v_2 - \Delta v_1)}{A_k} \right) \\ &\quad - \Delta u_2^{m+1} \Delta v_2^{n+1} {}_2F_1 \left([1, m+n+2], m+2, \frac{\Delta u_2(\Delta v_2 - \Delta v_1)}{A_k} \right) \Bigg], \end{aligned} \tag{8}$$

where $g_k^{(m,n)}$ is the m,n derivative of $g(u,v)$ with respect to u,v at the expansion vertex (u_1,v_1) for element k, (computed from the WLS expansion). ${}_2F_1$ is the generalized hypergeometric function which is singular when the third argument is one. This happens when $\Delta u_1 = \Delta u_2, \Delta v_1 = 0$, or $\Delta v_2 = 0$. The first condition can always be avoided by shifting the vertex ordering one step in a cyclic fashion for triangles where $\Delta u_1 = \Delta u_2$. To avoid the singularities with Δv_1 or Δv_2 equal to zero, accurate results are obtained by setting a (double precision) tolerance of 10^{-9} times the largest dimension of a triangle, where the argument is set to this value. Alternatively, the integral on the first line of Eq. (8) can be easily done term-by-term for each value of m and n up to some practical limit of p. Symbolic manipulation software can be used to write a series of simple functions for the coefficients, which is much faster to evaluate. As for Gaussian quadrature, the sums in Eq. (8) can be accumulated into an integration vector which is then multiplied with the vector of function values to get the integral over the body.

3 Results



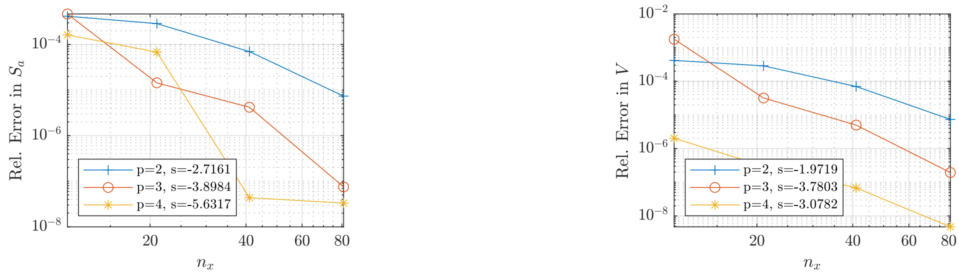
(a) The physical grid (not to scale). (b) The $u-v$ parameter space grid.

Figure 2: An example random point set grid over a Wigley hull.

To confirm the accuracy of the presented integration schemes, we compute the surface area and volume of the unit-length Wigley hull,

$$S_a = \int_S dS = 0.15013441092291333, \quad V = \left\{ -\int_S x_i n_i \, dS, \quad i = 1,2,3 \right\} = 13/4500. \tag{9}$$

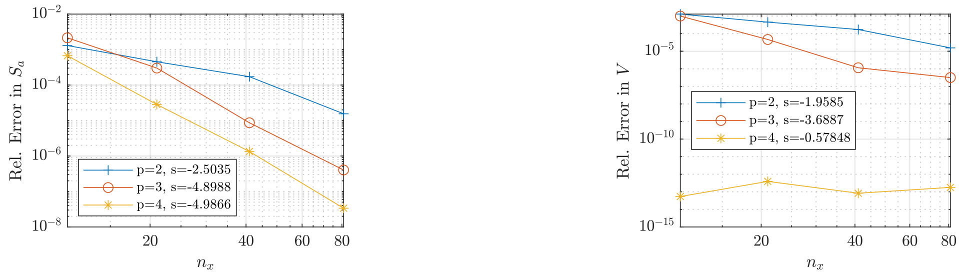
Figs. 3 & 4 show the convergence of the calculations with increasing number of points along the ship length, n_x, with $n_z \approx n_x/2$ vertical points and different orders of WLS interpolation, p. Here s indicates an estimate of the convergence rate for each curve. Despite the terrible grid quality, the convergence is reasonably uniform and consistent with order p to $p+1$. Perhaps more importantly, high accuracy is obtained using relatively few grid points.



(a) Convergence of the surface area.

(b) Convergence of the Volume.

Figure 3: Convergence of the Gaussian quadrature method on a random point set.



(a) Convergence of the surface area.

(b) Convergence of the Volume.

Figure 4: Convergence of the Taylor integration method on a random point set.

4 Conclusions

Two high-order methods for numerical surface integration on a scattered point set are developed and shown to provide highly accurate results. The methods can be applied to any hydrodynamic solver where the solution can be provided on an arbitrary selection of points over the hull surface. Additional examples using the method for hydrodynamic calculations will be presented at the Workshop.

References

[1] M. Amini-Afshar and H. B. Bingham. OceanWave3D-Seakeeping; an open source tool for predicting the seakeeping and added resistance of ships based on linear potential flow theory, 2023. `https://gitlab.gbar.dtu.dk/oceanwave3d/ow3d-seakeeping`.

[2] L. Brand. *Vector and Tensor Analysis*. John Wiley & Sons Inc., New York, 1947.

[3] W. D. Henshaw, D. W. Schwendeman, J. W. Banks, and K. K. Chand. Overture: an object-oriented toolkit for solving partial differential equations in complex geometries, 2023. `http://www.overtureframework.org/index.html`.

[4] C. T. Reddy and D. J. Shippy. Alternative integration formulae for triangular finite elements. *Int. J. for Numerical Methods in Engineering*, 17:133–153, 1981.

[5] O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu. *The Finite Element Method, its Basis and Fundamentals*. Elsevier Butterworth-Heinemann, 2005.