Trapped hydroelastic wave in the channel with a bottom step

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HIGHLIGHTS In this work, we investigate the interaction between a flexural-gravity wave propagating along the interface between an elastic sheet covering the liquid surface and a bottom step representing abrupt depth transitions. Within the framework of the potential flow model and the Cosserat theory of hyperelastic shells, we establish a fully nonlinear solution using the integral hodograph method to determine the higher harmonics of the interface shape

1 Introduction.

The nonlinear interaction of gravity waves with varying bathymetry is one of the mechanisms leading to the development of large amplitude waves [1, 2]. These extreme waves exhibit strongly nonlinear behavior and generate higher harmonics, observable as secondary crests in the troughs of the main waves [3]. Earlier studies focused on the transmission of regular waves over a step of infinite length using direct measurements [4]. Massel [5] developed the second-order theory and derived expressions for both the linear and second-order super-harmonic components, determining the reflection and transmission coefficients of a long wave traveling over an infinite step. Li and others [6] extended Massel's work to narrow-banded wave packets on the shallower side.

In contrast to previous studies, we consider a somewhat more complicated case that includes an elastic sheet covering the liquid surface. Our objective is to study a flexural-gravity wave propagating over the step on the bottom of the channel in still water. We employ the integral hodograph method to derive the complex velocity potential, which explicitly incorporates the velocity magnitude at the ice/liquid interface and the slope of the bottom. The coupling of the elastic sheet and liquid solutions is based on the condition of equal pressure at the interface, stemming from both flow dynamics and elastic sheet equilibrium. The entire problem is reduced to a system of nonlinear equations in the unknown velocity magnitude at the interface, which is then solved numerically.

2. Formulation of the problem.

We consider a two-dimensional steady flow in a channel with a fixed obstruction on the bottom and a step moving together with the liquid downstream. The channel is covered by an elastic plate modeling an ice sheet. We define a Cartesian coordinate system XY with the origin at the center of the fixed obstruction. The flow far upstream and downstream is assumed to be uniform with velocity U due to the motion of the bottom step with the same speed U. In the coordinate system attached to the moving bottom step, the same flow looks different. The liquid far from the step is at rest in both directions, but the obstruction moves upstream, generating a wave on the ice/liquid interface. A schematic diagram with the coordinate system attached to the fixed obstruction is shown in Figure 1a. The liquid is inviscid and incompressible, and the flow is assumed to be irrotational, allowing us to use a potential flow model. The obstruction has a characteristic length a, the thickness of the sheet is b, and the height of the step is h

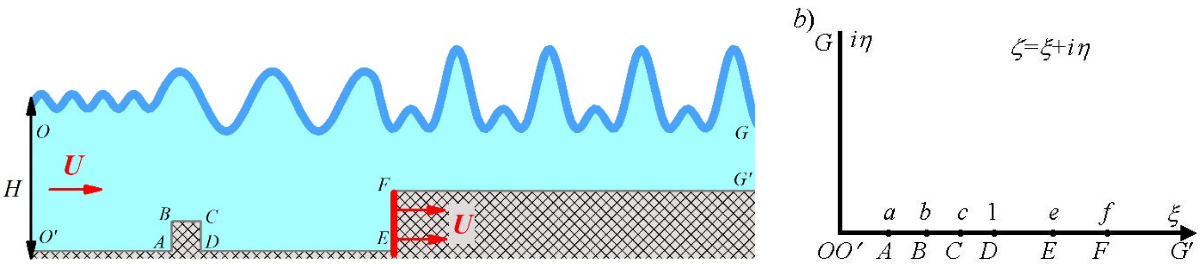

Figure 1: (a) physical plane; (b) parameter plane, or ζ- plane.

We introduce the complex velocity potential $W(Z) = \Phi(X,Y) + i\Psi(X,Y)$, comprising the velocity potential $\Phi(X,Y)$ and the stream function $\Psi(X,Y)$, where $Z = X + iY$. The boundary-value problem for the velocity potential can be written as follows:

(1) $\nabla^2\Phi = 0, \quad \nabla^2\Psi = 0,$ in the liquid domain;

(2) $\frac{\partial\Phi}{\partial Y} = \frac{\partial\Phi}{\partial X}\frac{\partial Y_b}{\partial X}, \quad \Psi = 0,$ on the bottom $Y_b(X)$ of the channel $OABCDE$ and FG.

(3) $\frac{\partial\Phi}{\partial X} = U, \quad \Psi = YU, \quad 0 \leq Y \leq h,$ on the moving side of the step EF.

(4) $\rho\frac{V^2}{2} + \rho gY + p_i = \rho\frac{U^2}{2} + \rho gH + p_\infty$, on the ice liquid interface.

Here, $V = |\nabla\Phi|$ is the velocity magnitude, $p_i = p_a$ for the free surface, and $p_i = p_{ice}(X)$ for the presence of the ice sheet; $p_\infty = p_a$ or $p_\infty = p_a + \rho_{ice}gb$ is the pressure at infinity; p_a is atmospheric pressure; ρ_{ice} is the ice density; b is the thickness of the ice sheet, and g is the gravity acceleration; the flow is steady; therefore, the value of the stream function at the interface is constant and equal to the flowrate across the channel $\Psi = UH$; on the step FG the stream function takes the value $\Psi = U(H - h)$. The elastic sheet is modelled using the Cosserat theory of hyperelastic shells

(5) $p_{ice} = D\left(\frac{d^2\kappa}{dX^2} + \frac{1}{2}\kappa^3\right) + p_a.$

where $D = Eb^3/[12(1-\upsilon^3)]$ is the flexural rigidity of the elastic sheet, κ is the curvature of the interface, E is Young's modulus, and ν is Poisson's ratio. Equation (5) corresponds to the inextensible and not prestressed ice sheet.

Complex potential.

We introduce the first quadrant as an auxiliary parameter plane, or $\zeta -$ plane (see figure 1b), and determine two functions, which are the derivative of the complex potential, $dw/d\zeta$, and the function of the complex velocity, dw/dz. Then, the interface OG can be obtained in parameter form as follows:

(6) $z(\zeta) = z_0 + \int_0^\xi \frac{dw}{d\zeta'}/\frac{dw}{dz}d\zeta'.$

We apply the integral hodograph method (the expression for solving a mixed boundary value problem for a complex function in the first quadrant [7, 8] to determine the expression for the complex velocity

(7) $\frac{dw}{dz} = v_0\sqrt{\frac{\zeta-a}{\zeta+a}\frac{\zeta+b}{\zeta-b}\frac{\zeta+c}{\zeta-c}\frac{\zeta-1}{\zeta+1}}exp\left[\frac{1}{\pi}\int_e^f\frac{d\beta}{d\xi}ln\left(\frac{\xi-\zeta}{\xi+\zeta}\right)d\xi - \frac{i}{\pi}\int_0^\infty\frac{d\ln v}{d\eta}ln\left(\frac{i\eta-\zeta}{i\eta+\zeta}\right)d\eta\right],$

where v_0 is the velocity magnitude at point O, $\beta(\xi)$ is the velocity direction on the vertical side of the bottom step EF, and $v = v(\eta)$ is the velocity magnitude on the interface OG', a, b, c, e, f are the parameters in the ζ −plane which are determined from the linear sizes of the obstruction and the bottom step.

The argument of the complex function $dw/d\zeta$ on the bottom is equal zero (because the normal component of the velocity is zero) excluding the part EF, where it is the function $\gamma(\xi)$. Using the integral formula for solving uniform boundary-value for a complex given its argument on the real and imaginary axes of the first quadrant [7, 8], we obtain the derivative of the complex potential in the form

(8) $\frac{dw}{d\zeta} = \frac{K}{\zeta}\sqrt{\frac{\zeta^2-f^2}{\zeta^2-e^2}}\exp\left[-\frac{1}{\pi}\int_e^f\frac{d\gamma}{d\xi}\ln\zeta^2 - \xi^2)d\xi\right],$

where K is the real constant.

It is seen from (8) that the complex potential w(ζ) has a logarithmic singularity at point $\zeta = 0$, or ζ exponentially grow as potential ϕ linearly increases. This causes difficulties in computations for distances $|X/H| > 5$. We resolve this singularity by eliminating variables ζ, ξ and η from equations (6) - (8) using the expressions:

$$
(9) \qquad \left.\begin{array}{lll}
\zeta = \exp(\pi\bar{w}/2), & -\infty < \bar{\phi} < \infty, & 0 \le \bar{\psi} \le 1, \\
\eta = \exp(\pi\bar{\phi}/2), & -\infty \le \bar{\phi} < \infty, & \bar{\psi} = 1, \\
\xi = \exp(\pi\bar{\phi}/2), & -\infty \le \bar{\phi} < \infty, & \bar{\psi} = 0.
\end{array}\right\}
$$

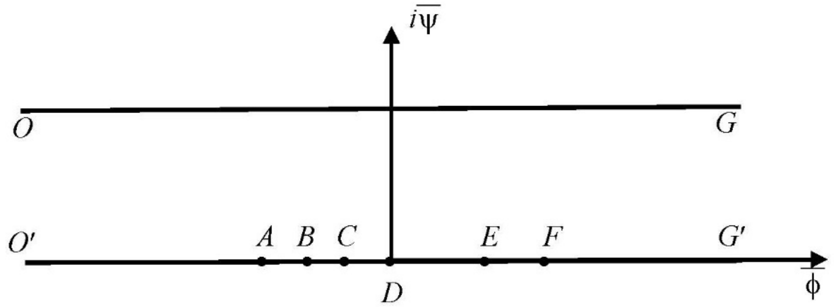

Figure 2. Parameter plane $\bar{w} = \bar{\phi} + i\bar{\psi} = \frac{2}{\pi}\ln\zeta$.

By using (9), we obtain the complex velocity (7) and the derivative of the complex potential (8) as the functions of the variable $\bar{w}$,

$$
(10) \quad \frac{dw}{dz} = v_0 \sqrt{\left(\frac{1-\exp[\pi(\bar{\phi}_A-\bar{w})]}{1+\exp[\pi(\bar{\phi}_A-\bar{w})]}\right)\left(\frac{1+\exp[\pi(\bar{\phi}_B-\bar{w})]}{1-\exp[\pi(\bar{\phi}_B-\bar{w})]}\right)\left(\frac{1+\exp[\pi(\bar{\phi}_C-\bar{w})]}{1-\exp[\pi(\bar{\phi}_C-\bar{w})]}\right)\left(\frac{1-\exp(-\pi\bar{w})}{1+\exp(-\pi\bar{w})}\right)}
$$
$$
\times \exp\left[\frac{1}{\pi}\int_{\bar{\phi}_E}^{\bar{\phi}_F} \frac{d\beta}{d\bar{\phi}} \ln\left(\frac{\exp[\pi(\bar{\phi}-\bar{w})]-1}{\exp[\pi(\bar{\phi}-\bar{w})]+1}\right) d\bar{\phi} - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\ln v}{d\bar{\phi}} \ln\left(\frac{\exp[\pi(\bar{\phi}+i-\bar{w})]-1}{\exp[\pi(\bar{\phi}+i-\bar{w})]+1}\right) d\bar{\phi}\right]
$$

$$
(11) \qquad \frac{dw}{d\bar{w}} = \frac{dw}{d\zeta}\frac{d\zeta}{d\bar{w}} = \frac{\pi K}{2}\sqrt{\frac{1-\exp[\pi(\bar{\phi}_F-\bar{w})]}{1-\exp[\pi(\bar{\phi}_E-\bar{w})]}} \exp\left[-\frac{1}{\pi}\int_{\bar{\phi}_E}^{\bar{\phi}_F} \frac{d\gamma}{d\bar{\phi}} \ln[\exp(\pi\bar{w}) - \exp(\pi\bar{\phi})] d\bar{\phi}\right]
$$

Finally, we obtain the derivative of the mapping function as the functions of $\bar{w}$

$$
(12) \qquad \frac{dz}{d\bar{w}} = \frac{dw}{d\bar{w}} \Big/ \frac{dw}{dz}.
$$

System of equations.
The governing equations (10) and (11) contain parameters $\bar{\phi}_A$, $\bar{\phi}_B$, $\bar{\phi}_C$, $\bar{\phi}_E$, $\bar{\phi}_F, K$ and two unknown functions $\beta(\xi)$ and $\gamma(\xi)$. The parameters $\bar{\phi}_A$, $\bar{\phi}_B, \bar{\phi}_C, \bar{\phi}_E, \bar{\phi}_F$ are determined using linear dimension of the obstruction and the step

$$
(13) \qquad \int_{a,b,c,e,f}^{1} \frac{ds}{d\bar{\phi}} d\bar{\phi} = S_{\{AD,BD,CD,ED,FD\}},
$$

where $\frac{ds}{d\bar{\phi}} = \left|\frac{dz}{d\bar{w}}\right|_{\bar{w}=\bar{\phi}}$, and $S_{\{AD,BD,CD,ED,FD\}}$ is the arclength from point D to points A, B, C, E, F.

The argument of the derivative of the mapping function along the vertical side of the step is equal to $\pi/2$. Taking the argument of (11), we obtain the relation between the angles γ, β on the vertical side of the step,

$$
(14) \qquad \arg\left(\frac{dz}{d\bar{w}}\right)_{\bar{w}=\bar{\phi}} = \arg\left(\frac{dw}{d\bar{w}}\right)_{\bar{w}=\bar{\phi}} - \arg\left(\frac{dw}{dz}\right)_{\bar{w}=\bar{\phi}} =
$$
$$
= \gamma(\bar{\phi}) + \beta(\bar{\phi}) = \pi/2, \qquad \bar{\phi}_E < \bar{\phi} < \bar{\phi}_F.
$$

The normal component of the velocity on the vertical side EF is $v_n = 1$, because the step moves with the same velocity as the liquid. The tangential component v_s is obtained as follows

$$(15) \qquad v_s(\xi) = Re\left(\frac{dw}{dz}\frac{dz}{ds}\right) = Re\left(\left.\frac{dw}{dz}\right|_{\overline{w}=\bar{\bar{\phi}}} i\right),$$

where $dz/ds = i$ on the vertical side EF. Then, using the definition of the angle γ, we obtain

$$(16) \qquad \gamma(\xi) = \arctan\frac{v_n}{v_s} = \arctan\left(\frac{1}{Re\left(dw/dz|_{\overline{w}=\bar{\phi}}\right)}\right).$$

The system of equations (13)–(14) and (16) allows us to determine the parameters $\bar{\phi}_A$, $\bar{\phi}_B$, $\bar{\phi}_C, \bar{\phi}_E, \bar{\phi}_F$ and the functions $\gamma(\bar{\phi})$ and $\beta(\bar{\phi})$. The function $v(\bar{\phi})$ is determined numerically. In discrete form, the solution is sought on two fixed sets of points: a set $-\bar{\phi}^* < \bar{\phi}_j < \bar{\phi}^*, j = 1,\ldots,N$ corresponding to the bottom of the channel and a set $-\bar{\phi}^* < \bar{\phi}_i < \bar{\phi}^*$, $i = 1,\ldots,M$ corresponding to the interface; both sets of points $\bar{\phi}_j$ and $\bar{\phi}_i$ monotonically increase. By applying the dynamic boundary condition (4) at the points $\bar{\phi}_k$, $k = 1,\ldots,\overline{K}$, we can obtain the following system of nonlinear equations

$$(17) \qquad G_k(\overline{V}) = c_{pk}(\overline{V}) - c_{pk}^{ice}(\overline{V}) = 0, \qquad k = 1,\ldots,\overline{K},$$

where $\overline{V} = (v_1, v_2, \ldots, v_{\overline{K}})^T$ is the vector of the unknown velocities v_k;

$$(18)-(19) \quad c_{pk}(\overline{V}) = 1 - v_k^2 - \frac{2[y_k(\overline{V})-1]}{F^2}; \qquad c_{pk}^{ice}(\overline{V}) = 2D\left[\left(\frac{d^2\kappa}{ds^2}\right)_k + \frac{1}{2}\kappa_k^3\right].$$

are the hydrodynamic pressure coefficient and the pressure coefficient due to the elastic sheet, respectively. The system of equations (17) is solved using Newton's method.

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