Interaction of flexural gravity waves with an ice sheet having variable geometry in the presence of current

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Highlights

- The flexural gravity wave interaction with an ice sheet having variable geometry has been examined.
- An ice sheet with a Gaussian oscillatory pattern is investigated in the presence of uniform current.
- Bragg resonating pattern is observed in reflection coefficient for very small values of spreading parameter.
- The peak due to resonance experiences a leftward shift with an increase in the depth Froude number and a rightward shift with an increase in the frequency of oscillation.
- This study provides a comprehensive understanding of the effects of current speed, the Bragg resonating phenomenon and the role of variable geometry in the ice sheet.

1 Introduction

Analysing the mathematical complexity of the interaction between elastic floating bodies and free-surface gravity waves in an intermediate water depth is a mathematically challenging problem for its important practical applications. Based on the existing data, approximately one-fifth of the Earth's oceans and seas are enveloped in ice. Thus, ice sheets are the most extensively researched elastic floating bodies, giving rise to flexural gravity waves when they interact with water waves. The presence of variable ice surface along the northern shore of Ellesmere Island, located in Nunavut, Canada [Nekrasov & MacAyeal (2023)], shows that the interaction between gravity waves and variable ice sheets requires extensive examination. There is also presence of currents in the ocean beneath the ice cover in several circumstances. These ocean currents can have a substantial impact on the wave field. Consequently, it is pertinent and practically significant to investigate the interaction between flexural gravity waves and an ice sheet with variable geometry in the presence of current, which is the primary goal of this study. This study could assist geologists and marine engineers in developing and maintaining ports and harbour infrastructure. This paper attempts to investigate the problem of interaction of flexural gravity waves with an ice sheet having variable geometry in the presence of current within the framework of linear water wave theory. The theoretical results are supported by the numerical computations for the real physical parameters involved.

2 Mathematical formulation

The horizontal wave motion beneath an elastic plate with variable geometry in an ocean of finite depth h, with uniform ocean current U along the x-directional flow is considered which is passing over an impermeable flat bottom. A Cartesian coordinate system (x, y) is used to formulate the physical problem, with y standing for the vertical coordinate and

x for the horizontal coordinate. The plate is considered to be extended infinitely as $x \to \pm\infty$. The floating thin elastic plate model [Fox & Squire (1991)] and potential flow theory for the motion of water waves are used for the mathematical formulation of the problem.

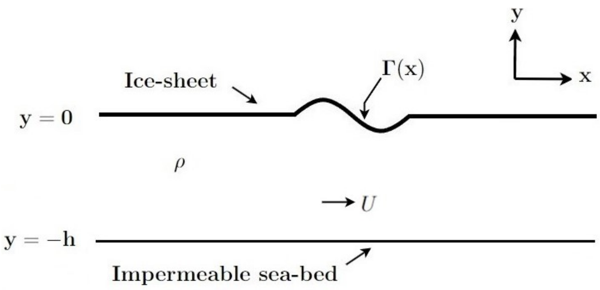


Figure 1: Schematic Diagram

Furthermore, the undulated plate has been expressed in the form of $y = P(x)$, where $P(x) = \varepsilon\Gamma(x)$, where $\Gamma(x)$ signifies the shape of variation in the plate, which is a differentiable function having compact support, i.e., $\Gamma(x) \to 0$ as $|x| \to \infty$, the nondimensional number $\varepsilon(\ll 1)$ indicates the smallness of the variation and $y = 0$ is the mean plate position. The fluid is considered to be inviscid and incompressible and flow is irrotational and time harmonic with frequency σ. The Boundary value problem (BVP) in non-dimensionalised form for velocity potential ϕ is given by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \text{ in the fluid region,} \tag{1}$$

$$\mathcal{L}\frac{\partial\phi}{\partial y} + \left(\mathrm{i}\sigma - F\frac{\partial}{\partial x}\right)^2\phi - \varepsilon\left\{\frac{d\Gamma(x)}{dx}\mathcal{L}\frac{\partial\phi}{\partial x} + \left(\frac{\partial\phi}{\partial x} + F\right)\mathcal{L}_\Gamma\frac{d\Gamma(x)}{dx}\right\} = 0, \text{ on } y = P(x), \tag{2}$$

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = -1, \tag{3}$$

where $\mathcal{L} = \left(D\frac{\partial^4}{\partial x^4} + Q\frac{\partial^2}{\partial x^2} + 1 - \delta_p\sigma^2\right)$, $\mathcal{L}_\Gamma = \left(D\frac{\partial^4}{\partial x^4} + Q\frac{\partial^2}{\partial x^2} + 1\right)$ with $D = \tilde{D}/\rho g h^4$, $Q = \tilde{Q}/\rho g h^2$, $\delta_1 = \delta/\rho h$, $\tilde{D} = Ed^3/[12(1-v^2)]$ denotes the flexural rigidity of the ice sheet; E Young's modulus; v Poisson's ratio; $\delta = \rho_1 d$ is the unit mass; d the ice-sheet thickness, which is considered to be very small; F is the depth Froude number; ρ and ρ_1 are the densities of fluid and ice sheet, respectively.
The elastic plate surface condition (2) can be written in the form

$$\begin{aligned}&\mathcal{L}\frac{\partial\phi}{\partial y} + \left(\mathrm{i}\sigma - F\frac{\partial}{\partial x}\right)^2\phi + \varepsilon\left\{\Gamma(x)\mathcal{L}\frac{\partial^2\phi}{\partial y^2} - \frac{d\Gamma(x)}{dx}\mathcal{L}\frac{\partial\phi}{\partial x}\right.\\ &\left.+\Gamma(x)\left(\mathrm{i}\sigma - F\frac{\partial}{\partial x}\right)^2\frac{\partial\phi}{\partial y} - \left(\frac{\partial\phi}{\partial x} + F\right)\mathcal{L}_\Gamma(x)\frac{d\Gamma(x)}{dx}\right\} = 0,\end{aligned} \quad \text{on } y = 0. \tag{4}$$

In addition, the radiation condition is given by

$$\phi(x,y) \sim \begin{cases}\phi_I(x,y) + \tilde{R}\phi_R(x,y), \ x \to -\infty,\\ \tilde{T}\phi_T(x,y), \ x \to \infty,\end{cases} \tag{5}$$

where ϕ_I, ϕ_R and ϕ_T are the progressive wave solution, reflected wave solution, and transmitted wave solution, respectively, with $\tilde{R}$ and $\tilde{T}$, the unknown coefficients associated with the reflected and transmitted waves, respectively, which will be examined in the following section.

3 Method of solution

Using perturbation technique, we can express ϕ, $\tilde{R}$ and $\tilde{T}$ as

$$\phi = \phi_0 + \varepsilon\phi_1 + \mathcal{O}(\varepsilon^2), \ \tilde{R} = \varepsilon R + \mathcal{O}(\varepsilon^2), \ \tilde{T} = 1 + \varepsilon T + \mathcal{O}(\varepsilon^2) \tag{6}$$

where $\phi_0 = e^{\mathrm{i}k_0 x}\mathfrak{h}(k_0, y)$, with $\mathfrak{h}(k_0, y) = \dfrac{\cosh k_0(y+1)}{\cosh k_0}$, where k_0 satisfies the dispersion relation

$$\mathfrak{D}(k) = 0; \tag{7}$$

where $\mathfrak{D}(k) = (\sigma - k_0 F)^2 - \mathcal{M}(k_0)k_0 \tanh(k_0)$, $\mathcal{M}(k) = \left(Dk^4 - Qk^2 + 1 - \delta_p \sigma^2\right)$, ϕ_1, R and T are the first order velocity potential, reflection coefficient and transmission coefficient respectively. Using equation (6) in equations (1), (2), (4) and (5), and comparing $\mathcal{O}(\varepsilon^1)$ terms, we will obtain the first order BVP. Solving the first order BVP using Fourier transform method, we obtain R and T as

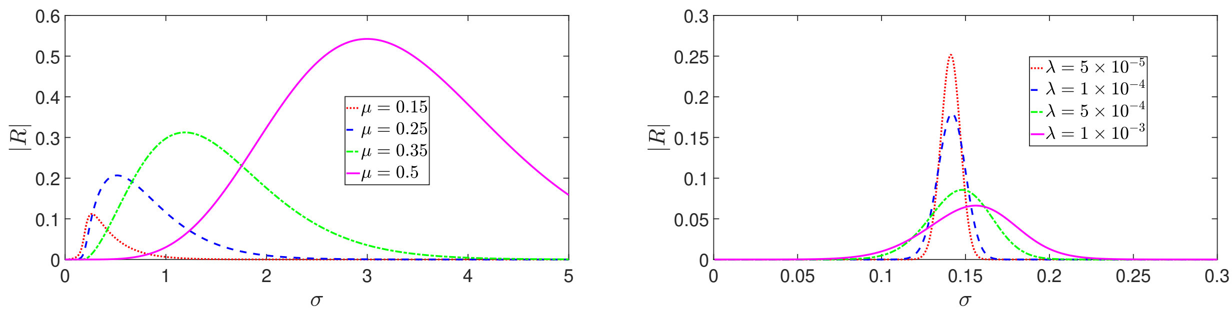
$$R = \mathrm{i}\frac{\overline{\beta}(-k_r)}{\mathfrak{D}'(-k_r)}, \text{ and } T = \mathrm{i}\frac{\overline{\beta}(k_t)}{\mathfrak{D}'(k_t)}, \tag{8}$$

$$\begin{aligned} \text{where } \overline{\beta}(\xi) = &\left[k_0(\sigma - k_0 F)^2 \tanh(k_0) - k_0 \mathcal{M}(k_0) - k_0(\xi - k_0)\mathcal{V}(\xi - k_0)\right] \overline{\Gamma}(\xi - k_0) \\ &+ \mathrm{i}\xi U \mathcal{V}(\xi)\overline{\Gamma}(\xi), \end{aligned} \tag{9}$$

$\mathcal{V}(y) = (Dy^4 - Qy^2 + 1)$, $\overline{\Gamma}(\xi) = \displaystyle\int_{-\infty}^{\infty} \Gamma(x)e^{-\mathrm{i}\xi x}\,\mathrm{d}x$, and k_r and k_t are the roots of dispersion relations $(\sigma + k_r F)^2 - \mathcal{M}(k_r)k_r \tanh(k_r) = 0$ and (7) respectively.

4 Numerical results

For the numerical computations, the values of $D = 4.554$, $Q = \sqrt{D}$ and $\delta_p = 0.089$ are kept fixed unless stated otherwise.



(a) (b)

Figure 2: Reflection coefficient $|R|$ versus σ for different (a) μ with $\lambda = 0.1$, $a = 0.01$ and (b) λ with $\mu = 0.1$ and $a = 0.01$.

Now consider a specific shape function of the variable geometry of the elastic plate as given by

$$\Gamma(x) = ae^{-\lambda x^2} \sin\left(2\pi\mu x\right), \quad -\infty < x < \infty, \ \lambda > 0, \tag{10}$$

where a being the amplitude of the oscillation, λ is the spreading parameter, and μ is the frequency of the oscillation.

Figure 2 illustrates the behaviour of first order reflection coefficient for different frequency parameter μ (Figure 2(a)) and different spreading parameter λ (Figure 2(b)) in the absence of current. Figure 2(a) depicts the formation of a single harmonic peak for all values of μ with the peak shifting towards right as the value of μ increases. Furthermore, it is noted that the magnitude of reflection is increasing significantly with an increase in the frequency of oscillation. This result provides significance for practical applications. Figure 2(b) illustrates that there is formation of smooth Bragg resonating pattern as the value of spreading parameter λ is decreased. Interestingly, the sharpness of the Bragg resonating peak increases as λ decreases. Perhaps this might be because the sinusoidal component of the plate variation dominates the Gaussian element as λ decreases, contributing to the formation of the Bragg resonating pattern.

Figure 3 demonstrates the behaviour of first order reflection coefficient with respect to different Froude numbers. It can be observed from Figure 3 that the magnitude of the reflection coefficient is decreasing with an increase in Froude number. Furthermore, as the value of the Froude number F increases, a leftward shift in the Bragg resonating peak is observed. The dispersion relation and wave modes are altered by currents, which may result in modifications to the interference patterns that support Bragg scattering.

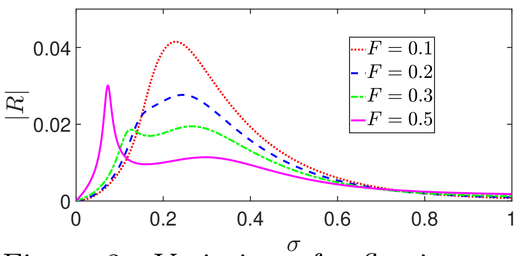


Figure 3: Variation of reflection coefficient for different Froude numbers F with $\lambda = 0.1$, $a = 0.01$ and $\mu = 0.1$.

5 Conclusion

The interaction of flexural gravity waves with an ice sheet having variable geometry in the presence of current has been studied. The associated boundary value problem is solved using perturbation technique followed by Fourier transform method. A particular type of shape function, known as Gaussian oscillatory, is used to analyse the variable geometry of the ice sheet. The reflection coefficient exhibits a uniform Bragg resonating pattern for very low values of the spreading parameter. An increase in Froude number results in a leftward shift in the Bragg resonating peak, while an increase in the frequency of oscillation results in a rightward shift. The present study provides a comprehensive analysis of the impact of current speed, the Bragg resonating phenomenon, and the significance of variation in the ice sheet.

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